ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-409 *Proposed by John Turner, University of Waikato, New Zealand*

Fibonacci-T Arithmetic Triangles

The following arithmetic triangle has many properties of special interest to Fibonacci enthusiasts.

							1							
						1	1	1						
					1	2	2	2	1					
				1	3	4	5	4	3	1				
			1	4	7	10	11	10	7	4	1			
		1	5	11	18	24	26	24	18	11	5	1		
	1	6	16	30	46	58	63	58	46	30	16	6	1	
1	7	22	47	81	116	143	153	143	116	18	47	22	7	1

. . . .

Denote the triangle by T, the i^{th} element in the n^{th} row by t_i^n, and the sum of elements in the n^{th} row by σ_n.

(i) Discover a rule to generate the next row from the previous rows.

(ii) Given your rule, prove the Fibonacci row-sum property, viz:

$$\sigma_n = 2\sum_{i=1}^{n-1} t_i^n + t_n^n = F_{2n}, \text{ for } n = 1, 2, \ldots,$$

where F_{2n} is a Fibonacci integer.

(iii) Discover and prove a remarkable functional property of the sequence of diagonal sequences, $\{d_i\}$:

$$\begin{aligned} d_1 &= 1 \quad 1 \quad 1 \quad 1 \quad 1 \ldots \\ d_2 &= 1 \quad 2 \quad 3 \quad 4 \quad 5 \ldots \\ d_3 &= 1 \quad 2 \quad 4 \quad 7 \quad 11 \ldots \\ d_4 &= 2 \quad 5 \quad 10 \quad 18 \quad 30 \ldots \\ d_5 &= 1 \quad 4 \quad 11 \quad 24 \quad 46 \ldots \end{aligned}$$

(iv) Discover another Fibonacci arithmetic triangle which has the same generating rule and other properties but with row-sums equal to F_{2n-1}, $n = 1, 2, \ldots$.

(v) Show how the numbers in the triangle are related to the dual-Zeckendorf theorem on integer representations, which states (see [1]) that every positive integer N has one and only one representation in the form

$$N = \sum_{1}^{k} e_i u_i,$$

where the e_i are binary digits and $e_i + e_{i+1} \neq 0$ for $1 \leqslant i < k$, and $\{u_i\} = 1, 2, 3, 5, \ldots$, the Fibonacci integers.

There are many interesting identities derivable from the triangle relating the t_i^n with themselves, with the natural numbers and Fibonacci integers, and with the binomial coefficients. The proposer offers a prize of US $25 for the best list of identities submitted.

A final remark is that Pascal-T (see [2] and [3]) and Fibonacci-T triangles can curiously be linked to a common source. They both may be derived from studies of binary words whose digits have the properties of the e_i in part (v) above.

References

1. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly* 3, no. 1 (1965):1-8.
2. S. J. Turner. "Probability via the Nth Order Fibonacci-T Sequence." *The Fibonacci Quarterly* 17, no. 1 (1979):23-28.
3. J. C. Turner. "Convolution Trees and Pascal-T Triangles." (Submitted to *The Fibonacci Quarterly*, 1986.)

H-410 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ \text{for } n \geqslant 2.$$

Prove or disprove that, for $n \geqslant 1$,

$$\int_0^1 F_n(x)dx = \frac{1}{n}(L_n - (-1)^n - 1).$$

H-411 *Proposed by Paul S. Bruckman, Fair Oaks, CA*

Define the simple continued fraction $\theta(a, d)$ as follows:

$$\theta(a, d) \equiv [a, a + d, a + 2d, a + 3d, \ldots],\ a \text{ and } d \text{ real},\ d \neq 0. \tag{1}$$

Find a closed form for $\theta(a, d)$.

SOLUTIONS

Acknowledgment Correction:

H-377, H-379, and H-382 were solved by S. Papastavridis, P. Siafarikas, and P. Sypsas; H-381 was not solved by P. Siafarikas or P. Sypsas.

A Complex Problem

H-386 *Proposed by Paul S. Bruckman, Fair Oaks, CA*
(Vol. 23, no. 2, May 1985)

Define the multiple-valued *Fibonacci function* ${}^mF : \mathbb{C} \to \mathbb{C}$ as follows:

1. $${}^mF(z) = \frac{1}{\sqrt{5}}(\exp Lz - \exp L'z), \quad z \in \mathbb{C},\ m \in \mathbb{Z},$$

where $L = \log \alpha$, $\alpha = \frac{1}{2}(1 + \sqrt{5})$, $L' = (2m + 1)i\pi - L$, and "log" denotes the principal logarithm.

a. Show that ${}^mF(n) = F_n$ for all integers m and n.

b. Prove the multiplication formula

2. $$\prod_{m=0}^{n-1} {}^mF\left(k + \frac{r}{n}\right) = 5^{-\frac{1}{2}(n-1)}F_{nk+r}, \text{ where } n, k, r \text{ are integers with } 0 < r < n.$$

c. With m fixed, find the zeros of ${}^mF$.

Solution by the propeser

Proof of (a):
$$\begin{aligned} {}^mF(n) &= 5^{-\frac{1}{2}}\{\exp(nL) - \exp[(2m + 1)ni\pi - nL]\} \\ &= 5^{-\frac{1}{2}}(\alpha^n - (-1)^{(2m+1)n}\alpha^{-n}) \\ &= 5^{-\frac{1}{2}}(\alpha^n - (-\alpha)^{-n}) = 5^{-\frac{1}{2}}(\alpha^n - \beta^n), \end{aligned}$$
where $\beta = \frac{1}{2}(1 - \sqrt{5})$; hence, ${}^mF(n) = F_n$. Q.E.D.

Proof of (b): Let $\omega = \exp(i\pi r/n)$. Then
$$\begin{aligned} \prod_{m=0}^{n-1} {}^mF(k + r/n) &= \prod_{m=0}^{n-1} 5^{-\frac{1}{2}}\{\exp(k + r/n)L \\ &\qquad - \exp[(2m + 1)(k + r/n)i\pi - (k + r/n)L]\} \\ &= 5^{-\frac{1}{2}n}\prod_{m=0}^{n-1}\{\alpha^{k+r/n} - (-1)^k\omega^{2m+1}\alpha^{-k-r/n}\} \\ &= 5^{-\frac{1}{2}n}\alpha^{nk+r}\prod_{m=0}^{n-1}\{1 - (-1)^k\omega^{2m+1}\alpha^{-2k-2r/n}\}. \end{aligned}$$

Since the solutions of the equation: $z^n = (-1)^r$ are given by $\omega, \omega^3, \omega^5, \ldots, \omega^{2n-1}$, it follows that, for all x,
$$(1 - x\omega)(1 - x\omega^3) \cdots (1 - x\omega^{2n-1}) = 1 - (x\omega)^n = 1 - (-1)^r x^n.$$
Therefore, setting $x = (-1)^k\alpha^{-2k-2r/n}$, we see that
$$\begin{aligned} \prod_{m=0}^{n-1} {}^mF(k + r/n) &= 5^{-\frac{1}{2}n}\alpha^{nk+r}\{1 - (-1)^{nk+r}\alpha^{-2nk-2r}\} \\ &= 5^{-\frac{1}{2}n}(\alpha^{nk+r} - \beta^{nk+r}) = 5^{-\frac{1}{2}(n-1)}F_{nk+r}. \quad \text{Q.E.D.} \end{aligned}$$

Note that setting $r = 0$ in (2) yields F_k^n [using (a)].

Solution of (c): ${}^mF(z) = (2/\sqrt{5})\exp(m + \tfrac{1}{2})i\pi z \quad \sinh z\theta_m$, where

$$\theta_m = \tfrac{1}{2}(L - L') = \log \alpha - (m + \tfrac{1}{2})i\pi.$$

Since exp uz vanishes for *no* complex u and z, the zeros of ${}^mF$ are precisely the zeros of sinh $z\theta_m$, namely, ${}^mF(z_{r,m}) = 0$, where

$$z_{r,m} = ri\pi/\theta_m = r \cdot z_{1,m} = \frac{-r(m + \frac{1}{2})\pi^2 + \pi rLi}{L^2 + (m + \frac{1}{2})^2\pi^2}. \quad \text{Q.E.D.}$$

NOTE: Given m, ${}^mF$ is one of the Riemann sheets which extend the Fibonacci numbers to the complex domain.

Also solved by L. Kuipers.

Non Residual

H-387 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 23, no. 2, May 1985)

Let $\{w_n\}_{n=0}^{\infty}$ be a second-order linear integral recurrence defined by the recursion relation

$$w_{n+2} = aw_{n+1} + bw_n,$$

where $b \neq 0$. Show the following:

(i) If p is an odd prime such that $p \nmid b$ and $w_1^2 - w_0w_2$ is a quadratic nonresidue of p, then

$$p \nmid w_{2n} \quad \text{for any } n \geq 0.$$

(ii) If p is an odd prime such that $(-b)(w_1^2 - w_0w_2)$ is a quadratic nonresidue of p, then

$$p \nmid w_{2n+1} \quad \text{for any } n \geq 0.$$

(iii) If p is an odd prime such that $-b$ is a nonzero quadratic residue of p and $w_1^2 - w_0w_2$ is a quadratic nonresidue of p, then

$$p \nmid w_n \quad \text{for any } n \geq 0.$$

Solution by the proposer

We first note that

$$w_n^2 - w_{n-1}w_{n+1} = (-b)^{n-1}(w_1^2 - w_0w_2) \tag{1}$$

for $n \geq 1$. This identity can be proven by induction using the recursion relation defining $\{w_n\}$. We now prove parts (i), (ii), and (iii).

(i) Suppose $p|w_{2n}$ for some $n \geq 0$. Then by (1),

$$w_{2n+1}^2 - w_{2n}w_{2n+1} \equiv w_{2n+1}^2 - 0 \equiv (-b)^{2n}(w_1^2 - w_0w_2) \pmod{p}.$$

However, this is contradicted by the fact that $w_1^2 - w_0w_2$ is a quadratic nonresidue of p and $-b$ is a nonzero residue of p. The result follows.

(ii) Suppose $p|w_{2n+1}$ for some $n \geq 0$. Then by (1),

$$\begin{aligned} w_{2n+1}^2 - w_{2n+1}w_{2n+3} &\equiv w_{2n+2}^2 - 0 \equiv (-b)^{2n+1}(w_1^2 - w_0w_2) \\ &\equiv (-b)^{2n}[(-b)(w_1^2 - w_0w_2)] \pmod{p}. \end{aligned}$$

This is a contradiction, since $(-b)(w_1^2 - w_0w_2)$ is a quadratic nonresidue of p and the product of a nonzero quadratic residue and a quadratic nonresidue is a quadratic nonresidue. Hence, assertion (ii) must hold.

(iii) This follows immediately from parts (i) and (ii). First, by (i), p cannot divide w_{2n} for any $n \geqslant 0$, since $p \nmid b$ and $w_1^2 - w_0w_2$ is a quadratic nonresidue of p. Also, by (ii), p cannot divide w_{2n+1} for any $n \geqslant 0$, since $(-b)(w_1^2 - w_0w_2)$ is a quadratic nonresidue of p. This again follows, because the product of a nonzero quadratic residue and a quadratic nonresidue is a quadratic nonresidue. Thus, $p \nmid w_n$ for any $n \geqslant 0$, and we are done.

Also solved by P. Bruckman, L. Kuipers, and T. White.

Across the Digraph!

H-388 *Proposed by Piero Filipponi, Rome, Italy*
(Vol. 23, no. 2, May 1985)

This problem arose in the determination of the diameter of a class of locally restricted digraphs [1].

For a given integer $n \geqslant 2$, let $P_1 = \{p_{1,1}, p_{1,2}, \ldots, p_{1,k_1}\}$ be a nonempty (i.e., $k_1 \geqslant 1$) increasing sequence of positive integers such that $p_{1,k_1} \leqslant n - 1$. Let $P_2 = \{p_{2,1}, p_{2,2}, \ldots, p_{2,k_2}\}$ be the increasing sequence containing all nonzero distinct values given by $p_{1,i} + p_{1,j} \pmod{n}$ $(i, j = 1, 2, \ldots, k_1)$. In general let $P_h = \{p_{h,1}, p_{h,2}, \ldots, p_{h,k_h}\}$ be the increasing sequence containing all nonzero distinct values given by $p_{h-1,i} + p_{1,j} \pmod{n}$ $(i = 1, 2, \ldots, k_{h-1}, j = 1, 2, \ldots, k_1)$. Furthermore, let B_m $(m = 1, 2, \ldots)$ be the increasing sequence containing all values given by

$$\bigcup_{j=1}^{m} P_j.$$

Find, in terms of $n, p_{1,1}, \ldots, p_{1,k_1}$, the smallest integer t such that

$$B_t = \{1, 2, \ldots, n - 1\}.$$

Remark: The necessary and sufficient condition for t to exist (i.e., to be finite) is given in [1]:

$$\gcd(n, p_{1,1}, \ldots, p_{1,k_1}) = 1.$$

In such a case we have $1 \leqslant t \leqslant n - 1$. It is easily seen that

$$k_1 = 1 \Longleftrightarrow t = n - 1$$
$$k_1 = n - 1 \Longleftrightarrow t = 1;$$

furthermore, it can be conjectured that either $t = n - 1$ or $1 \leqslant t \leqslant [n/2]$.

Reference

1. P. Filipponi. "Digraphs and Circulant Matrices." *Ricerca Operativa*, no. 17 (1981):41-62.

An Example

$$n = 8 \quad P_1 = \{3, 5\} \rightarrow B_1 = \{3, 5\}$$
$$P_2 = \{2, 6\} \rightarrow B_2 = \{2, 3, 5, 6\}$$
$$P_3 = \{1, 3, 5, 7\} \rightarrow B_3 = \{1, 2, 3, 5, 6, 7\}$$
$$P_4 = \{2, 4, 6\} \rightarrow B_4 = \{1, 2, 3, 4, 5, 6, 7\};$$ hence, we have $t = 4$.

Comments (not a solution) by Paul S. Bruckman, Fair Oaks, CA

The proposer's conjecture may be refined to the following conjecture:

$$1 \leqslant t \leqslant \left[\frac{n-2}{k_1}\right] + 1. \tag{1}$$

This *seems* to be true, but a more exact expression eluded me, as did the proof of (1). I first conjectured that $t = t(n, k_1)$ was a function solely of n and k_1, given by:

$$t = \left[\frac{n-2}{k_1}\right] + 1. \tag{2}$$

Unfortunately, (2) is false; the first counter-example occurs with $n = 8$, $k_1 = 2$. If we take $P_1 = (5, 7)$, then $P_2 = (2, 4, 6)$, $P_3 = (1, 3, 5, 7)$, so $t = 3$ in this case. On the other hand, if $P_1 = (1, 2)$, then $P_2 = (2, 3, 4)$, $P_3 = (3, 4, 5, 6)$, $P_4 = (4, 5, 6, 7)$, so $t = 4$ in this case.

Thus t, in general, depends on P_1 as well as on n and k_1. It is conceivable, however, that (2) would hold, provided some additional constraints on P_1 are specified. Note that the expression given in (2) produces the correct values of t (for $n \geqslant 2$) if $k_1 = 1$ or $k_1 = n - 1$. It seems likely that the problem is more difficult than the proposer originally intended, at least in its general form, and that the true formula for $t = t(n, k_1, P_1)$ is more complicated than some concise expression such as indicated in (2).

Waiting for Success

H-389 *Proposed by Andreas N. Philippou, University of Patras, Greece (Vol. 23, no. 3, August 1985)*

Show that

$$F_{n+2}^{(n-i)} = 2^n - 2^i(1 + i/2) \qquad (n \geqslant 2i + 1)$$

for each nonnegative integer i, where $F_{n+2}^{(n-i)}$ is the $n + 2$ Fibonacci number of order $n - i$ [1] and $F_3^{(1)} = 1$.

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.

Solution by S. Papastavridis, P. Siafarikas, & P. Sypsas, U. of Patras, Greece

Setting $(n - i) = k$ and $(n + 2) = m$, the problem becomes

$$F_m^{(k)} = 2^{m-2} - 2^{m-k-2}\left(1 + \frac{m-2-k}{2}\right)$$

or

$$F_m^{(k)} = 2^{m-2} - 2^{m-k-3}(m - k) \tag{1}$$

for $k + 2 \leqslant m \leqslant 2k + 2$.

We shall prove (1) for $k + 2 \leqslant m \leqslant 2k + 2$. From here on, we suppress (k), since it is the same throughout. So we write F_m instead of $F_m^{(k)}$.

In the paper of Philippou and Muwafi ([1], p. 29, Lemma 2.1), it is proved that the sequence F_m satisfies the following recursion:

$$F_m = \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m = 1, 2 \\ 2F_{m-1} & \text{if } k + 1m \geqslant 3 \\ 2F_{m-1} - F_{m-k-1} & \text{if } m \geqslant k + 2 \end{cases}$$

This clearly implies that the generating function $\sum_{m=0}^{\infty} F_m t^m$ equals

$$F(t) = \sum_{m=0}^{\infty} F_m t^m = \frac{t - t^2}{1 - 2t + t^{k+1}}$$

We are going to expand this generating function. Binomial expansion is all we need. Thus, we have:

$$F(t) = \frac{t - t^2}{1 - 2t + t^{k+1}} = \frac{t - t^2}{1 - t(2 - t^k)} \doteq (t - t^2)\sum_{i=0}^{\infty} t^i(2 - t^k)^i$$

$$= (t - t^2)\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}2^{i-j}t^{kj+i}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}2^{i-j}t^{kj+i+1} - \sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^j\binom{i}{j}2^{i-j}t^{kj+i+2}$$

(in the first summation we set $m = kj + i + 1$, and in the second summation we set $m = kj + i + 2$)

$$= \sum_{m=1}^{\infty}\left(\sum_{j=0}^{(m-1)/(k+1)}(-1)^j\binom{m - 1 - kj}{j}2^{m-1-(k+1)j}\right)t^m$$

$$= \sum_{m=2}^{\infty}\left(\sum_{j=0}^{(m-2)/(k+1)}(-1)^j\binom{m - 2 - kj}{j}2^{m-2-(k+1)j}\right)t^m.$$

Thus, since F_m is the coefficient of t^m in the expansion of $F(t)$, we get:

$$F_m = \sum_{j=0}^{(m-1)/(k+1)}(-1)^j\binom{m - 1 - kj}{j}2^{m-(k+1)j-1} - \sum_{j=0}^{(m-2)/(k+1)}(-1)^j\binom{m - 2 - kj}{j}2^{m-(k+1)j-2}, \text{ for } m \geqslant 2. \tag{2}$$

Formula (2) is a general closed expression of F_m. Let us look at the special case that we have with the conditions

$(m - 1)/(k + 1) < 2$ and $(m - 2)/(k + 1) \geqslant 1$,

which is equivalent to

$k + 3 \leqslant m \leqslant 2k + 2$.

In this case, the index j in the summations of (2) takes only the values $j = 0$ and $j = 1$. So, we obtain (for this case)

$$F_m = 2^{m-1} - (m - 1 - k)2^{m-k-2} - 2^{m-2} + (m - k - 2)2^{m-k-3}$$

$$= 2^{m-2} - 2^{m-k-2}\left(\frac{m - k}{2}\right)$$

$$= 2^{m-2} - 2^{m-k-3}(m - k),$$

which is exactly what we had to prove. The remaining case of $k + 2 = m$ is deduced similarly. The case of $F_3^{(1)}$ is obvious.

Reference

1. A. N. Philippou & A. A. Muwafi. "Waiting for the k^{th} Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly* 20, no. 1 (1982):28-32.

Also solved by P. Bruckman, B. Poonen, and the proposer.

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