CONVERGENCE OF TRIBONACCI DECIMAL EXPANSIONS

RICHARD H. HUDSON
University of South Carolina, Columbia, SC 29208

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1. INTRODUCTION

Let F_i denote the Fibonacci sequence defined by

$$F_1 = F_2 = 1,\ F_i = F_{i-2} + F_{i-1},\ \text{for } i \geqslant 3;$$

that is, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... In 1953 Fenton Stancliff [5] observed that

$$\sum_{i=1}^{\infty} 10^{-(i+1)} F_i = \frac{1}{89}. \tag{1}$$

Since 1953 a number of authors including Wlodarski [8], Brousseau [1], Kohler [3], Winans [7], Long [5], Hudson and Winans [2], and Pin-Yen Lin [4] have investigated the convergence of Fibonacci decimal expansions,

$$\sum_{i=1}^{\infty} 10^{-k(i+1)} F_{\alpha i},\ \alpha \geqslant 1.$$

C. F. Winans first observed that

$$\sum_{i=1}^{\infty} 10^{-(i+1)} F_{2i}$$

appears to converge to 1/71 employing decimal approximation, since

$$\frac{1}{71} = .014084507\ldots$$

and

$$\begin{aligned} \sum_{i=1}^{10} 10^{-(i+1)} F_{2i} = \ & .01 \\ + \ & .0003 \\ + \ & .00008 \\ + \ & .000021 \\ + \ & .0000055 \\ + \ & .00000144 \\ + \ & .000000377 \\ + \ & .0000000987 \\ + \ & .00000002584 \\ + \ & \underline{.000000006765} \\ & .010408448305 \end{aligned} \tag{2}$$

Convergence of (2) to 1/71 was proved in [2], as were

$$\sum_{i=1}^{\infty} 10^{-(i+1)} F_{2i} = \frac{2}{59} \quad \text{and} \quad \sum_{i=1}^{\infty} 10^{-(i+1)} F_{3i} = \frac{3}{31}.$$

The purpose of this paper is to prove an analogous conjecture of Winans for tribonacci decimal expansions and to generalize this result to obtain convergents in cases where Winans found that decimal approximation failed to give even a clue to the correct convergent. As in the Fibonacci case, the convergents include coefficients that involve a fascinating, though more complicated, tribonacci-like recurrence relation; see Theorem 2 in Section 3.

2. PROOF OF WINAN'S CONJECTURE

Let T_i denote the tribonacci sequence defined by $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and

$$T_i = T_{i-3} + T_{i-2} + T_{i-1}, \quad i \geqslant 3; \tag{3}$$

that is, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, ... Employing decimal approximation, Winans conjectured the following theorem which we now prove.

Theorem 1: Let T_i be defined as in (3). Then

$$\sum_{i=1}^{\infty} 10^{-k(i+2)} T_i = \frac{1}{10^{3k} - 10^{2k} - 10^k - 1}. \tag{4}$$

Proof: Define $f(z)$ by

$$f(z) = \sum_{i=1}^{\infty} T_i z^i \tag{5}$$

and note that since $T_1 = T_2$, $T_3 = T_1 + T_2$, and $T_i = T_{i-1} + T_{i-2} + T_{i-3}$ for $i \geqslant 4$, we have

$$\begin{aligned}(1 - z - z^2 - z^3)f(z) &= (1 - z - z^2 - z^3)(T_1 z + T_2 z^2 + \cdots)\\ &= T_1 z + (T_2 - T_1)z^2 + (T_3 - T_2 - T_1)z^3 + (T_4 - T_3 - T_2 - T_1)z^4 + \cdots\\ &\qquad + (T_n - T_{n-1} - T_{n-2} - T_{n-3})z^n + \cdots\\ &= T_1 z + (T_2 - T_1)z^2 + (T_3 - T_2 - T_1)z^3 = z.\end{aligned}$$

Therefore,

$$f(z) = \sum_{i=1}^{\infty} T_i z^i = \frac{z}{1 - z - z^2 - z^3}. \tag{6}$$

Since $|1 - z - z^2 - z^3| \geqslant 1 - |z| - |z^2| - |z^3| > 0$ if $|z| \leqslant 1/2$, the function $f(z)$ is analytic in the disc $\{z \in C : |z| \leqslant 1/2\}$. Consequently, its power series expansion is absolutely convergent for all z with $|z| \leqslant 1/2$ and (6) holds if we replace z by any complex number with modulus less than or equal to 1/2.

In particular, if we let $z = 10^{-k}$ with $k \geqslant 1$, we obtain

$$\sum_{i=1}^{\infty} T_i 10^{-ki} = \frac{10^{-k}}{1 - 10^{-k} - 10^{-2k} - 10^{-3k}} = \frac{10^{2k}}{10^{3k} - 10^{2k} - 10^k - 1}, \tag{7}$$

completing the proof of the conjecture of Winans.

Remark: Define an n-ary Fibonacci sequence by the recurrence relation

$$T_{i,n} = T_{i-n-1,n} + T_{i-n-2,n} + \cdots + T_{i-1,n} > n \geqslant 2, \; i \geqslant n. \tag{8}$$

Using the same method given in the proof of Theorem 1, one obtains:

$$\sum_{i=1}^{\infty} 10^{-k(i+n-1)} T_{i,n} = \frac{1}{10^{nk} - 10^{(n-1)k} - \cdots - 1}. \tag{9}$$

This result was conjectured by Winans for tetrabonacci and pentabonacci expansions.

Numerical Examples: Analogous to (1), observed by Stancliff, we have from (7),

$$\sum_{i=1}^{\infty} 10^{-(i+1)} T_i = \frac{1}{1000 - 100 - 10 - 1} = \frac{1}{889}.$$

Moreover, by (9), we have

$$\sum_{i=1}^{\infty} 10^{-(i+1)} T_{i,4} = \frac{1}{10000 - 1000 - 100 - 10 - 1} = \frac{1}{8889},$$

and, in general (with the dots denoting $n - 1$ eights),

$$\sum_{i=1}^{\infty} 10^{-(i+1)} T_{i,n} = 888\ldots89$$

for an n-ary Fibonacci decimal expansion.

3. GENERALIZATION OF WINAN'S CONJECTURES

For $\alpha \geqslant 2$, Winans was unable to formulate a conjecture for the correct convergents for $\Sigma 10^{-ki} T_{\alpha i}$ even for $k = 1$, $\alpha = 2$. Once one establishes the correct convergent as we will in Theorem 3 of this section, one observes that $\Sigma 10^{-i} T_{2i}$ does converge fairly rapidly to 110/689. Indeed,

$$\begin{aligned}\sum_{i=1}^{10} 10^{-i} T_{\alpha i} = \quad & .1 \\ +\ & .04 \\ +\ & .013 \\ +\ & .0044 \\ +\ & .00149 \\ +\ & .000504 \\ +\ & .0001705 \\ +\ & .00005768 \\ +\ & .000019513 \\ \hline & .159641693\end{aligned}$$

and $\dfrac{110}{689} = .159651699\ldots$.

First, we require a theorem involving a recurrence relation for tribonacci numbers which is interesting in itself and essential to the goal of determining all convergents of

$$\sum_{i=1}^{\infty} 10^{-ki} T_{\alpha i}, \quad k \geqslant 1, \ \alpha \geqslant 1.$$

Theorem 2: Let $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and let $T_i = T_{i-3} + T_{i-2} + T_{i-1}$ for $i \geqslant 3$.

Define sequences $\{a_i\}$ and $\{b_i\}$ by

$$a_i = a_{i-1} + a_{i-2} + a_{i-3} \text{ for } i \geqslant 4;\ a_1 = 1,\ a_2 = 3,\ a_3 = 7, \tag{10}$$

and

$$b_i = b_{i-1} + b_{i-2} + b_{i-3} \text{ for } i \geqslant 4;\ b_1 = b_2 = 1,\ b_3 = -5. \tag{11}$$

For every positive integer $\alpha \geqslant 1$,

$$T_{3\alpha+i} = a_\alpha T_{2\alpha+i} + b_\alpha T_{\alpha+i} + T_i;\ i \geqslant 0. \tag{12}$$

Proof: Let β_1, β_2, and β_3 be the distinct complex roots of $z^3 - z^2 - z - 1 = 0$ so that

$$(z - \beta_1)(z - \beta_2)(z - \beta_3) = z^3 - z^2 - z - 1. \tag{13}$$

Then there are constants u_1, u_2, and u_3 such that

$$T_i = u_1\beta_1^i + u_2\beta_2^i + u_3\beta_3^i \text{ for every } i \geqslant 0. \tag{14}$$

Define

$$A_\alpha = \beta_1^\alpha + \beta_2^\alpha + \beta_3^\alpha,\ B_\alpha = -[\beta_1^\alpha\beta_2^\alpha + \beta_1^\alpha\beta_3^\alpha + \beta_2^\alpha\beta_3^\alpha],\ C_\alpha = (\beta_1\beta_2\beta_3)^\alpha.$$

Now, it is easily checked that

$$(\beta_1^\alpha)^3 - (\beta_1^\alpha + \beta_2^\alpha + \beta_3^\alpha)(\beta_1^\alpha)^2 + [(\beta_1\beta_2)^\alpha + (\beta_1\beta_3)^\alpha + (\beta_2\beta_3)^\alpha]\beta_1^\alpha - (\beta_1\beta_2\beta_3)^\alpha$$

$$= \beta_1^{3\alpha} - \beta_1^{3\alpha} - \beta_2^\alpha\beta_1^{2\alpha} - \beta_3^\alpha\beta_1^{2\alpha} + \beta_2^\alpha\beta_1^{2\alpha} + \beta_3^\alpha\beta_1^{2\alpha} + (\beta_2\beta_3\beta_1)^\alpha - (\beta_1\beta_2\beta_3)^\alpha = 0,$$

and similarly for β_2^α and β_3^α, so that β_1^α, β_2^α, and β_3^α are the roots of the equation

$$z^3 - A_\alpha z^2 - B_\alpha z - C_\alpha = 0.$$

Using (14), we obtain

$$T_{i+3\alpha} - A_\alpha T_{i+2\alpha} - B_\alpha T_{i+\alpha} - C_\alpha T_i = 0. \tag{15}$$

From (13), it follows that

$$A_i = A_{i-1} + A_{i-2} + A_{i-3} \text{ for every } i \geqslant 1 \tag{16}$$

[Since, for $j = 1, 2, 3$, $\beta_j^i = \beta_j^{i-1} + \beta_j^{i-2} + \beta_j^{i-3} \Longleftrightarrow \beta_j^{i-3}(\beta_j^3 - \beta_j^2 - \beta_j - 1) = 0$] and clearly

$$C_i = 1 \text{ for every } i \geqslant 0. \tag{17}$$

In particular, (13) implies that $\beta_1\beta_2\beta_3 = 1$, so that

$$B_\alpha = -[\beta_1^{-\alpha} + \beta_2^{-\alpha} + \beta_3^{-\alpha}]. \tag{18}$$

Replacing z by z^{-1} in (13), we obtain

$$\left(\frac{1}{z} - \beta_1\right)\left(\frac{1}{z} - \beta_2\right)\left(\frac{1}{z} - \beta_3\right) = \frac{1}{z^3} - (\beta_1 + \beta_2 + \beta_3)\left(\frac{1}{z^2}\right) + (\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)\left(\frac{1}{z}\right) - \beta_1\beta_2\beta_3,$$

so that $\beta_1 + \beta_2 + \beta_3 = 1$ and $\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 = -1$.

On the other hand, we have, as $\beta_1\beta_2\beta_3 = 1$,

$$\left(\frac{1}{z} - \frac{1}{\beta_1}\right)\left(\frac{1}{z} - \frac{1}{\beta_2}\right)\left(\frac{1}{z} - \frac{1}{\beta_3}\right) = \frac{1}{z^3} - \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} + \frac{1}{\beta_3}\right)\left(\frac{1}{z^2}\right) + \left(\frac{1}{\beta_3}\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) + \frac{1}{\beta_1\beta_2}\right)\left(\frac{1}{z}\right) - 1$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} - 1,$$

so that $1/\beta_1$, $1/\beta_2$, and $1/\beta_3$ are roots of $(1/z)^3 + (1/z)^2 + (1/z) - 1 = 0$. However, $\beta_2\beta_3 = 1/\beta_1$, $\beta_1\beta_3 = 1/\beta_2$, and $\beta_1\beta_2 = 1/\beta_3$, so we have

$$B_i = -\left[\left(\frac{1}{\beta_3}\right)^i + \left(\frac{1}{\beta_2}\right)^i + \left(\frac{1}{\beta_1}\right)^i\right].$$

Consequently,

$$B_i = -B_{i-1} - B_{i-2} + B_{i-3} \text{ for } i \geqslant 3 \tag{19}$$

if and only if

$$\begin{aligned}-&\left[\left(\frac{1}{\beta_3}\right)^i + \left(\frac{1}{\beta_2}\right)^i + \left(\frac{1}{\beta_1}\right)^i\right] \\ &= \left[\left(\frac{1}{\beta_3}\right)^{i-1} + \left(\frac{1}{\beta_2}\right)^{i-1} + \left(\frac{1}{\beta_1}\right)^{i-1}\right] + \left[\left(\frac{1}{\beta_3}\right)^{i-2} + \left(\frac{1}{\beta_2}\right)^{i-2} + \left(\frac{1}{\beta_1}\right)^{i-2}\right] \\ &\qquad - \left[\left(\frac{1}{\beta_3}\right)^{i-3} + \left(\frac{1}{\beta_2}\right)^{i-3} + \left(\frac{1}{\beta_1}\right)^{i-3}\right].\end{aligned}$$

But this is equivalent, since $\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 = -1$, to

$$-\left(\left(\left(\frac{1}{\beta_3}\right)^{i-3} + \left(\frac{1}{\beta_2}\right)^{i-3} + \left(\frac{1}{\beta_1}\right)^{i-3}\right)\left(\sum_{j=1}^{3}\left(\left(\frac{1}{\beta_j}\right)^3 + \left(\frac{1}{\beta_j}\right)^2 + \left(\frac{1}{\beta_j}\right) - 1\right)\right)\right) = 0,$$

which is true in view of the fact that $1/\beta_1$, $1/\beta_2$, $1/\beta_3$ are roots of

$$\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right) - 1 = 0.$$

Finally, checking initial values, we observe from (16), (17), and (19) that $a_\alpha = A_\alpha$, $b_\alpha = B_\alpha$, and $C_\alpha = 1$ for every $\alpha \geqslant 1$ completing the proof of Theorem 2.

Using Theorem 2, we can now easily establish our main result, from which convergents of all tribonacci expansions of the form $\sum 10^{-ki} T_{\alpha i}$, $\alpha \geqslant 1$, $k \geqslant 1$, may be calculated. Clearly, this contains Theorem 1 as the special case $\alpha = k = 1$. However, we note that the proof of the following theorem does not appear to generalize trivially to n-ary Fibonacci expansions, $n > 3$, because of its dependence on Theorem 2.

Theorem 3: Let $\{T_i\}$, $\{a_i\}$, and $\{b_i\}$ be defined as in Theorem 2. Then

$$\sum_{i=1}^{\infty} 10^{-ki} T_{\alpha i} = \frac{T_\alpha \cdot 10^{2k} + (T_{2\alpha} - a_\alpha T_\alpha) \cdot 10^k}{10^{3k} - a_\alpha \cdot 10^{2k} - b_\alpha \cdot 10^k - 1} \tag{20}$$

iff the denominator is nonnegative.

Proof: Define $F(z)$ by

$$F(z) = \sum_{i=1}^{\infty} T_{\alpha i} z^i, \ \alpha \geqslant 1, \tag{21}$$

and observe that

$$\begin{aligned}&(1 - a_\alpha z - b_\alpha z^2 - z^3)(T_\alpha z + T_{2\alpha} z^2 + T_{3\alpha} z^3 + \cdots) \\ &= T_\alpha z + (T_{2\alpha} - a_\alpha T_\alpha) z^2 + (T_{3\alpha} - T_{2\alpha} a_\alpha - T_\alpha b_\alpha) z^3 + \text{terms of higher degree.}\end{aligned}$$

Using (12), it is easily seen that the coefficients of all powers of z greater than 2 vanish. Hence,

$$F(z) = \frac{T_\alpha z + (T_{2\alpha} - a_\alpha T_\alpha)z^2}{1 - a_\alpha z - b_\alpha z^2 - z^3}.$$

Let γ_1^α, γ_2^α, and γ_3^α be the roots of

$$1 - a_\alpha z - b_\alpha z^2 - z^3 = 0. \tag{22}$$

We begin by showing that exactly one of the roots of (22) is real. Indeed, it suffices to consider the case $\alpha = 1$. For, assume that one of γ_1^α, γ_2^α, γ_3^α, say γ_1, is nonreal and that all of γ_1^α, γ_2^α, γ_3^α are real. Clearly, $Q(\gamma_1^\alpha)$ is a proper subfield of $Q(\gamma_1)$, so that $\deg(Q(\gamma_1^\alpha/Q)) < 3$ and divides 3; that is, it is 1. Consequently, γ_1^α is an algebraic integer lying in Q. Indeed, it is a unit because it is a root of $1 - a_\alpha z - b_\alpha z^2 - z^3 = 0$, so that $\gamma_1^\alpha = \pm 1$. Thus, γ_1 is a root of unity, which is clearly impossible.

It is now easy to show that (22) has exactly one positive real root when $\alpha = 1$. Let $f(z) = 1 - z - z^2 - z^3$ and observe that $f'(z) = -1 - 2z - 3z^2 < 0$ for all real z since $f''(z) = -2 - 6z = 0$ only if $z = -1/3$ and $f'''(-1/3) < 0$ so that $f'(z)$ has a maximum at $z = -1/3$. However, $f'(-1/3) < 0$ so that $f(z)$ is decreasing for all real z and, since $f(1/2) > 0$ and $f(1) < 0$, it is clear that $f(z) = 0$ has one real root z, $(1/2) < z < 1$, and so must have two nonreal roots which are conjugate pairs.

Now, let $h(z)$ be the polynomial defined by

$$h(z) = T_\alpha z + (T_{2\alpha} - a_\alpha T_\alpha)z^2.$$

Then, applying partial fractions, we have, as $\alpha = 1$,

$$\begin{aligned} F(z) &= \frac{u_1 z}{\gamma_1 - z} + \frac{u_2 z}{\gamma_2 - z} + \frac{u_3 z}{\gamma_3 - z} \\ &= z\left(\frac{u_1}{\gamma_1}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma_1}\right)^n + \frac{u_2}{\gamma_2}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma_2}\right)^n + \frac{u_3}{\gamma_3}\sum_{n=0}^{\infty}\left(\frac{z}{\gamma_3}\right)^n\right). \end{aligned}$$

This converges if $\left|\frac{z}{\gamma_1}\right| < 1$, $\left|\frac{z}{\gamma_2}\right| < 1$, and $\left|\frac{z}{\gamma_3}\right| < 1$.

Now the denominator of $F(z)$ can be written as

$$(\gamma_1 - z)(\gamma_2 - z)(\gamma_3 - z)$$

and if we let γ_1 be the real root between 1/2 and 1 and note that

$$(\gamma_2 - z)(\gamma_3 - z) > 0,$$

since γ_2 and γ_3 are complex conjugates, we see that, for real z,

$$1 - z - z^2 - z^3 > 0 \text{ if and only if } \gamma_1 - z > 0 \text{ or } z < \gamma_1. \tag{23}$$

Clearly, then, as $\gamma_1\gamma_2\gamma_3 = 1$ and $|\gamma_2| = |\gamma_3| = (1/\sqrt{\gamma_1}) > 1$, we also have

$$|z| < |\gamma_2| \quad \text{and} \quad |z| < |\gamma_3|,$$

completing the proof.

Example 1: Let $k = 3$ and let $\alpha = 8$. Then, by Theorem 3,

$$\sum_{i=1}^{\infty} 10^{-3i} T_{8i} = \frac{44 \cdot 10^6 + 4 \cdot 10^3}{10^9 - 131 \cdot 10^6 + 3 \cdot 10^3 - 1} = \frac{44{,}004{,}000}{869{,}002{,}999}.$$

Note that this fraction is approximately equal to .050637... and that with

$T_8 = 44$, $T_{16} = 5768$, $T_{24} = 755476$,

we have

$$\sum_{i=1}^{\infty} 10^{-3i} T_{8i} = \begin{array}{r} .044 \\ +\ .005768 \\ +\ .000755476 \\ \hline .050523476 \end{array}$$

so that the series converges quite rapidly for $k = 3$ although it does not converge at all for $k = 2$.

Example 2: Listed in the table below are the convergents of

$$\sum_{i=1}^{\infty} 10^{-ki} T_{\alpha i} \text{ for } k = 1, 2, 3 \text{ and } \alpha \leqslant 4.$$

	$k = 1$	$k = 2$	$k = 3$
$\alpha = 1$	$\frac{100}{889}$	$\frac{10{,}000}{989{,}899}$	$\frac{1{,}000{,}000}{998{,}998{,}999}$
$\alpha = 2$	$\frac{110}{689}$	$\frac{10{,}100}{969{,}899}$	$\frac{1{,}001{,}000}{996{,}998{,}999}$
$\alpha = 3$	$\frac{190}{349}$	$\frac{19{,}900}{930{,}499}$	$\frac{1{,}999{,}000}{993{,}004{,}999}$
$\alpha = 4$	None	$\frac{40{,}000}{889{,}499}$	$\frac{4{,}000{,}000}{988{,}994{,}999}$

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