ELEMENTARY PROBLEMS AND SOLUTIONS

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DEFINITIONS

The Fibonacci numbers \( F_n \) and the Lucas numbers \( L_n \) satisfy
\[
F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1
\]
and
\[
L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.
\]

PROBLEMS PROPOSED IN THIS ISSUE

B-598 Proposed by Herta T. Freitag, Roanoke, VA

For which positive integers \( n \) is \( (2L_n, 2^{2n} - 3, L_{2n} - 1) \) a Pythagorean triple? For which of these \( n \)'s is the triple primitive?

B-599 Proposed by Herta T. Freitag, Roanoke, VA

Do B-598 with the triple now \( (2L_n, 2^{2n} + 1, 2^{2n} + 3) \).

B-600 Proposed by Philip L. Mana, Albuquerque, NM

Let \( n \) be any positive integer and \( m = n^3 - n \). Prove that \( F_n \) is an integral multiple of 30290.

B-601 Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Let \( A_n,k = (F_n + F_{n+1} + \cdots + F_{n+k-1})/k \). Find the smallest \( k \) in \( \{2, 3, 4, \ldots\} \) such that \( A_n,k \) is an integer for every \( n \) in \( \{0, 1, 2, \ldots\} \).

B-602 Proposed by Paul S. Bruckman, Fair Oaks, CA

Let \( H_n \) represent either \( F_n \) or \( L_n \).

(a) Find a simplified expression for \( \frac{1}{H_n} - \frac{1}{H_{n+1}} - \frac{1}{H_{n+2}} \).

(b) Use the result of (a) to prove that
\[
\sum_{n=1}^{N} \frac{1}{F_n} = 3 + 2 \sum_{n=1}^{N} \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}}.
\]
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B-603 Proposed by Paul S. Bruckman, Fair Oaks, CA

Do the Lucas analogue of B-602(b).

SOLUTIONS

Downrounded Square Roots

B-574 Proposed by Valentina Bakinova, Rondout Valley, NY

Let \( a_1, a_2, \ldots \) be defined by \( a_1 = 1 \) and \( a_{n+1} = \lfloor \sqrt{a_n} \rfloor \), where \( a_n = a_1 + a_2 + \cdots + a_n \) and \( \lfloor x \rfloor \) is the integer with \( x - 1 < \lfloor x \rfloor < x \). Find \( a_{100}, a_{1000}, a_{10000}, \) and \( a_{100000} \).

Solution by L. A. G. Dressel, University of Reading, England

Starting with \( a_1 = 1 \), we have \( a_2 = a_3 = a_4 = 1 \) and \( a_5 = 4 \). Suppose now that, for some integer \( h, h \geq 2 \), we have \( a_i = h^2 \). Then, since \( (h+1)^2 = h^2 + 2h + 1 \), we obtain

\[
a_{t+1} = a_{t+2} = a_{t+3} = h \quad \text{and} \quad a_{t+3} = (h+1)^2 + h - 1;
\]

further,

\[
a_{t+4} = a_{t+5} = h + 1 \quad \text{and} \quad a_{t+5} = (h+2)^2 + h - 2,
\]

and continuing as long as \( j \leq h \), \( a_{t+2j+1} = (h+j)^2 + h - j \), so that for \( j = k \) we obtain \( a_{t+2h+1} = (2h)^2 \).

Since \( a_4 = 2^2 \), it follows that whenever \( a_n \) is a perfect square it is of the form \( 2^{2i} (i = 0, 1, 2, \ldots) \), and that if

\[
a_{t_i} = 2^{2i} \quad \text{and} \quad a_{t_i+1} = 2^{2(i+1)},
\]

then \( t_i+1 = a_i + 2^{i+1} + 1 \).

Since \( s_1 = 1, t_0 = 1 \), and we can show that

\[
t_i = 2^{i+1} + i - 1, \quad \text{for} \quad i = 0, 1, 2, \ldots.
\]

To find \( a_{100} \) and \( a_{1000} \): we have \( t_5 = 64 + 4 = 68 \), so that \( s_{68} = (32)^2 \),

\[
e_{99} = (32 + 15)^2 + 32 - 15, \quad a_{100} = 47, \quad e_{100} = (47)^2 + 64 = 2273.
\]

To find \( a_{1000} \) and \( a_{10000} \): \( t_8 = 2^8 + 7 = 519 \) and \( s_{519} = (256)^2 \),

\[
s_{999} = (256 + 239)^2 + 256 - 239, \quad a_{999} = a_{1000} = 495
\]

and

\[
s_{10000} = (256 + 240)^2 + 256 - 240 = (496)^2 + 16 = 246032.
\]

Also solved by Charles Ashbacher, Paul S. Bruckman, Piero Filipponi, L. Kuipers, J. Suck, M. Wachtel, and the proposer.

Summing Products

B-575 Proposed by L. A. G. Dressel, Reading, England

Let \( R_n \) and \( S_n \) be sequences defined by given values \( R_0, R_1, S_0, S_1 \) and the recurrence relations \( R_{n+1} = rR_n + tR_{n-1} \) and \( S_{n+1} = sS_n + tS_{n-1} \), where \( r, s, t \) are constants and \( n = 1, 2, 3, \ldots \). Show that

\[
(r + s) \sum_{k=1}^{n} R_k S_k t^{n-k} = (R_{n+1}S_n + R_nS_{n+1}) - t^n(R_2S_0 + R_0S_2).
\]
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Solution by J. Suck, Essen, Germany

This identity may be hard to dream up but is easy to prove by induction:

For \( n = 1 \), the left-hand side is \((r + s)R_1S_1\), and the right-hand side is

\[
(rR_1 + tR_6)S_1 + R_1(sS_1 + tS_2) - t(R_1S_6 + R_6S_1),
\]
i.e., both are the same.

For the step from \( n \) to \( n + 1 \), we have to show that

\[
t(R_{n+1}S_n + R_nS_{n+1}) + (r + s)R_{n+1}S_{n+1}
= (rR_{n+1} + tR_n)S_{n+1} + R_{n+1}(sS_{n+1} + tS_n),
\]
which, after a little sorting, is seen to be true.


Product of Three Fibonacci Numbers

B-576 Proposed by Herta T. Freitag, Roanoke, VA

Let \( A = L_{2m+3(n+1)} + (-1)^m \). Show that \( A \) is a product of three Fibonacci numbers for all positive integers \( m \) and \( n \).

Solution by Lawrence Somer, Washington, D.C.

We prove the more general result that, if \( r \geq 1 \), then

\[
L_{2r+1} + (-1)^r = 5F_rF_{r+1} = F_r^2F_{r+1}.
\]

Note that, if \( 2r + 1 = 2m + 3(4n + 1) \), then

\[
m \equiv r + 1 \pmod{2} \quad \text{and} \quad (-1)^m = (-1)^{r+1}.
\]

By the Binet formulas and using the fact that \( \alpha \beta = -1 \),

\[
5F_rF_{r+1} = 5[(\alpha^n - \beta^n)/\sqrt{5}][(\alpha^{n+1} - \beta^{n+1})/\sqrt{5}]
= \alpha^{2r+1} + \beta^{2r+1} - (\alpha \beta)^r(\alpha + \beta)
= L_{2r+1} - (-1)^rL_1 = L_{2r+1} + (-1)^{r+1},
\]
and we are done.


Difference of Squares

B-577 Proposed by Herta T. Freitag, Roanoke, VA

Let \( A \) be as in B-575. Show that \( 4A/5 \) is a difference of squares of Fibonacci numbers.

Solution by Bob Priellip, University of Wisconsin-Oshkosh, WI

Let \( m \) and \( n \) be arbitrary positive integers. We shall show that
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\[ 4A/5 = F_m^2 + 6n + 3 - F_m + 6n. \]  
\[ \text{(a)} \]

In our solution to B-576, we establish that
\[ A = 5F_m + 6n + 2F_m + 6n + 1. \]
Thus,
\[ 4A/5 = 4F_m + 6n + 2F_m + 6n + 1. \]

But it is known that \( 4F_k F_{k-1} = F_{k+1} - F_{k-2} \) [see (I.38) on p. 59 of Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969), so (a) follows.


Zeckendorf Representation for \([aF]\)

B-578 Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy

It is known (Zeckendorf's theorem) that every positive integer \( N \) can be represented as a finite sum of distinct nonconsecutive Fibonacci numbers and that this representation is unique. Let \( a = (1 + \sqrt{5})/2 \) and \([x]\) denote the greatest integer not exceeding \( x \). Denote by \( f(N) \) the number of \( F \)-addends in the Zeckendorf representation for \( N \). For positive integers \( n \), prove that \( f([aF_n]) = 1 \) if \( n \) is odd.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

It suffices to show that, for each positive integer \( n \), \([aF_{2n-1}]\) is a Fibonacci number. We shall show that,

for each positive integer \( n \), \([aF_{2n-1}] = F_{2n}\).

Let \( n \) be an arbitrary positive integer, and let \( b = (1 - \sqrt{5})/2 \). It is known that, for each positive integer \( k \), \( aF_k = F_{k+1} - b^k \) [see p. 34 of Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969). So \( aF_{2n-1} = F_{2n} - b^{2n-1} = F_{2n} - (-b)^{2n-1} \). Since \( 0 < -b < 1 \) and \( 0 < (-b)^{2n-1} < 1 \), it follows that \([aF_{2n-1}] = F_{2n}\).


Zeckendorf Representation, Even Case

B-579 Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy

Using the notation of B-578, prove that \( f([aF_n]) = n/2 \) when \( n \) is even.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Let \( n \) be an arbitrary positive integer. We shall show that the Zeckendorf representation for \([aF_{2n}]\) is \( F_2 + F_3 + F_5 + \cdots + F_{2n} \), which implies the required result.

Let \( b = (1 - \sqrt{5})/2 \). It is known that
\[ aF_{2n} = F_{2n+1} - b^{2n} \]
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[see p. 34 of Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. (Boston: Houghton-Mifflin, 1969). Since $0 < b^2 < 1$, $0 < b^{2n} < 1$. It follows that]
\[ [aF_{2n}] = F_{2n+1} - 1. \]
But
\[ F_{2n+1} - 1 = F_2 + F_4 + F_6 + \cdots + F_{2n} \]
by (Iₖ) (Ibid., p. 56). Hence, the Zeckendorf representation for $[aF_{2n}]$ is
\[ F_2 + F_4 + F_6 + \cdots + F_{2n} \]
completing our solution.


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