ON THE DERIVATIVES OF COMPOSITE FUNCTIONS

CLEMENT FRAPPIER
Université de Montréal, Montréal, Quebec H3C 3J7, Canada

(Submitted August 1985)

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1 Let f, g be functions sufficiently differentiable. Put $G(z) = f(z^z)$, where $z^z := \exp(z \ln z)$ ($\exp t := e^t$, $\ln 1 = 0$). If f is the identity function, i.e., if $G(z) = z^z$, then (see [7], p. 110)

$$G^{(m)}(1) = \begin{vmatrix} 1 & -1 & 0 & \dots 0 \\ (-1)^2 0! & \binom{1}{1} & -1 & \dots 0 \\ (-1)^3 1! & (-1)^2 0!\binom{2}{1} & \binom{2}{2} & \dots 0 \\ \dots & \dots & \dots & \dots \\ (-1)^{m-1}(m-3)! & (-1)^{m-2}(m-4)!\binom{m-2}{1} & (-1)^{m-3}(m-5)!\binom{m-2}{2} & \dots -1 \\ (-1)^m (m-2)! & (-1)^{m-1}(m-3)!\binom{m-1}{1} & (-1)^{m-2}(m-4)!\binom{m-1}{2} & \dots \binom{m-1}{m-1} \end{vmatrix} \tag{1}$$

for $m = 1, 2, 3, \ldots$. A particular case of a result obtained in this article shows that (1) may be replaced by

$$G^{(m)}(1) = \sum_{k=1}^{m} \sum_{\ell=1}^{k} (-1)^{k+m} S_1(m, k) \ell^{k-\ell} \binom{k}{\ell}, \tag{2}$$

where $S_1(m, k)$ is the sequence of Stirling numbers of the first kind, which may be defined by

$$S_1(m, 1) = (m - 1)!,$$

$$S_1(m, m) = 1,$$

and

$$S_1(m, k) = (m - 1)S_1(m - 1, k) + S_1(m - 1, k - 1), \quad 1 < k < m.$$

Let us consider the sequence $\omega(m, k, j)$ defined, for $0 \leqslant j \leqslant k$, $1 \leqslant k \leqslant m$, in the following way:

$$j!\omega(m, k, j) := \binom{m}{k - j} \sum_{s=0}^{j} (-1)^s \binom{j}{s} (k - s)^{m-k+j}. \tag{3}$$

We have

$$\omega(m, k, 0) = \binom{m}{k} k^{m-k},$$

$$\omega(m, m, j) = \binom{m}{j}$$

$$\left(\text{since } \sum_{s=0}^{j} (-1) \binom{j}{s} (m - s) = j!; \text{ note that } s\binom{j + 1}{s} = (j + 1)\binom{j}{s - 1}\right)$$

and (see [3], II, p. 38) $\omega(m, k, k) = S(m, k)$, the sequence of Stirling numbers of the second kind, which may be defined by

$$S(m, 1) = S(m, m) = 1$$

and

$$S(m, k) = kS(m - 1, k) + S(m - 1, k - 1), \ 1 < k < m.$$

That kind of generalization of Stirling numbers has already been considered by Carlitz ([1]; see also [2] and [4]). In fact, we have (see [1], II, p. 243)

$$\omega(m, k, j) = (-1)^{k+m}\binom{m}{k - j}R(m - k + j, j, -k),$$

where

$$\sum_{m=0}^{\infty} \sum_{j=0}^{m} R(m, j, \lambda)\frac{x^m y^j}{m!} = \exp(\lambda x + y(e^x - 1)), \ \lambda \in \mathbb{R}.$$

The combinatorial aspect of the sequence $R(m, j, \lambda)$ and other related numbers have been studied in the aforesaid articles. We want, here, to give some complements. To begin, we state the following theorem.

Theorem 1: Suppose that $G(z)$ is defined as above; we have

$$G^{(m)}(z) = \sum_{k=1}^{m} \sum_{\ell=1}^{k} \sum_{r=1}^{\ell} \sum_{s=0}^{\ell} (-1)^{k+m} S_1(m, k)S(\ell, r)\omega(k, \ell, s) z^{rz+\ell-m}(\ln z)^s f^{(r)}(z^z). \tag{4}$$

If $f(z) \equiv z$, then $G(z) = z^z$ and (4) becomes

$$G^{(m)}(z) = \sum_{k=1}^{m} \sum_{\ell=1}^{k} \sum_{s=0}^{\ell} (-1)^{k+m} S_1(m, k)\omega(k, \ell, s) z^{z+\ell-m}(\ln z)^s; \tag{5}$$

we obtain (2) with $z = 1$.

While proving (4), we shall obtain some identities relating two differential operators, denoted by $f_m^{(3)}$, $f_m^{(4)}$, and defined by

$$f_0^{(3)} := f, \ f_1^{(3)}(z) := \exp\left(\frac{f'(z)}{f(z)}\right), \ f_m^{(3)} := (f_{m-1}^{(3)})_1^{(3)}, \ m > 1, \tag{6}$$

and

$$f_0^{(4)} := f, \ f_1^{(4)}(z) := \exp\left(\frac{zf'(z)}{f(z)}\right), \ f_m^{(4)} := (f_{m-1}^{(4)})_1^{(4)}, \ m > 1. \tag{7}$$

We shall in fact consider two well-known operators, denoted here by $f_m^{(1)}$, $f_m^{(2)}$, and defined by

$$f_0^{(1)} := f, \ f_1^{(1)}(z) := f'(z), \ f_m^{(1)} := (f_{m-1}^{(1)})_1^{(1)}, \ m > 1, \tag{6'}$$

and

$$f_0^{(2)} := f, \ f_1^{(2)}(z) := zf'(z), \ f_m^{(2)} := (f_{m-1}^{(2)})_1^{(2)}, \ m > 1. \tag{7'}$$

Those operators have been studied for a very long time. The operator f_1 is the ordinary derivative of f; it is easy to verify that

$$f_m^{(2)}(z) = \sum_{k=1}^{m} S(m, k) z^k f^{(k)}(z).$$

Of course $\ln f_1^{(3)}$ is nothing but the logarithmic derivative of f. The operator $\ln f_1^{(4)}$ is useful in geometric function theory; for example, a function $f(z)$, holomorphic in the unit disk, is called starlike (see [6], p. 46) if

$$|f_1^{(4)}(z)| \geqslant 1$$

in that disk.

1.2 A classical formula of Faa Di Bruno ([3], I, p. 148; [5], p. 177) says that if $h(z) := f(g(z))$ then

$$h^{(m)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (g^{(j)}(z))^{k_j} \cdot f^{(k)}(g(z)) \tag{8}$$

where $\pi(m, k)$ means that the summation is extended over all nonnegative integers $k_1, \ldots, k_m$ such that $k_1 + 2k_2 + \cdots + mk_m = m$ and $k_1 + k_2 + \cdots + k_m = k$; we have put

$$c(k_1, \ldots, k_m) := \frac{m!}{k_1! \ldots k_m!(1!)^{k_1} \ldots (m!)^{k_m}}.$$

Formula (8) is equivalent to

$$\ln h_m^{(3)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (g^{(j)}(z))^{k_j} \cdot \ln f_k^{(3)}(g(z)). \tag{8'}$$

It can be proved in several ways; a simple proof is contained in [8]. We can prove the next theorem using only the principle of mathematical induction.

Theorem 2: If $h(z) := f(g(z))$, then we have the identities

$$h_m^{(2)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (g_j^{(2)}(z))^{k_j} \cdot f_k^{(1)}(g(z)) \tag{9}$$

and

$$\ln h_m^{(4)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (\ln g_j^{(4)}(z))^{k_j} \cdot \ln f_k^{(4)}(g(z)). \tag{9'}$$

Formula (9′) may also be written in the form

$$H_m^{(2)}(z) = \sum_{k=1}^{m} \sum_{\pi(m,k)} c(k_1, \ldots, k_m) \prod_{j=1}^{m} (g_j^{(2)}(z))^{k_j} \cdot f_k^{(2)}(e^{g(z)}), \tag{9''}$$

where $H(z) := f(\exp(g(z)))$.

1.3 If f^{-1} denotes the inverse function of f [i.e.,

$$f(f^{-1}(z)) \equiv f^{-1}(f(z)) \equiv z],$$

then (see [3], I, p. 161), for $m = 2, 3, 4, \ldots$,

$$(f^{-1})_m^{(1)}(z) \tag{10}$$

$$= \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} \frac{(-1)^k (m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \prod_{j=2}^{m} (f^{(j)}(f^{-1}(z)))^{k_j} \cdot (f'(f^{-1}(z)))^{-m-k},$$

where $\pi_1(m, k)$ means that the summation is extended over all nonnegative integers $k_2, \ldots, k_m$ such that $2k_2 + \cdots + mk_m = m + k - 1$ and $k_2 + \cdots + k_m = k$. Here,

$$c_1(k_1, \ldots, k_m) := c(0, k_2, \ldots, k_m).$$

The same kind of reasoning which could be used to prove (9) or (9′) will help us to verify the following theorem.

Theorem 3: If f^{-1} denotes the inverse function of f, then the following identities are valid for $m = 2, 3, 4, \ldots$:

$$(f^{-1})_m^{(2)} = \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} \frac{(-1)^k(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \cdot \prod_{j=2}^{m} (\ln f_j^{(3)}(f^{-1}(z)))^{k_j} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k}; \tag{11}$$

$$\ln(f^{-1})_m^{(3)}(z) = \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} \frac{(-1)^k(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \cdot \prod_{j=2}^{m} (f_j^{(2)}(f^{-1}(z)))^{k_j} \cdot (f_1^{(2)}(f^{-1}(z)))^{-m-k}; \tag{11'}$$

$$\ln(f^{-1})_m^{(4)}(z) = \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} \frac{(-1)^k(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \cdot \prod_{j=2}^{m} (\ln f_j^{(4)}(f^{-1}(z)))^{k_j} \cdot (\ln f_1^{(4)}(f^{-1}(z)))^{-m-k}. \tag{11''}$$

It is to be noted that (11′) may be obtained from (11″) by replacing $f(z)$ by $\exp f(z)$: also, if we replace $f(z)$ by $f(e^z)$ in (11) , then we obtain (11″). The distinction between formulas (8) and (9) and formulas (10) and (11) is also to be observed. Finally, while the identity

$$\ln\left(f(z)^{g(z)}\right)_m^{(3)} = \sum_{k=0}^{m} \binom{m}{k} g^{(m-k)}(z) \ln f_k^{(3)}(z)$$

is nothing but the Leibnitz formula, we have

$$\ln\left(f(z)^{g(z)}\right)_m^{(4)} = \sum_{k=0}^{m} \binom{m}{k} g_{m-k}^{(2)}(z) \ln f_k^{(4)}(z)$$

or, what is the same thing (see [5], p. 222):

$$(f(z)g(z))_m^{(2)} = \sum_{k=0}^{m} \binom{m}{k} f_k^{(2)}(z) g_{m-k}^{(2)}(z).$$

2. COMPLEMENTARY RESULTS

It follows from the recurrence relations for Stirling's numbers that:

Lemma 1: We have, for $m = 1, 2, 3, \ldots$,

$$f_m^{(2)}(z) = \sum_{k=1}^{m} S(m, k) z^k \cdot f_k^{(1)}(z) \tag{12}$$

and

$$z^m f_m^{(1)}(z) = \sum_{k=1}^{m} (-1)^{k+m} S_1(m, k) \cdot f_k^{(2)}(z). \tag{12'}$$

To obtain (4), we shall also need the following lemma.

Lemma 2: The sequence $\omega(m, k, j)$, defined by (3), satisfies the following recurrence relation:

$$\omega(m, 1, 0) = m, \ \omega(m, m, j) = \binom{m}{j} \quad (0 \leqslant j \leqslant m),$$
$$\omega(m, k, k) = S(m, k) \quad (1 \leqslant k \leqslant m),$$

and

$$\omega(m+1, k, 0) = k\omega(m, k, 0) + \omega(m, k-1), 0) + \omega(m, k, 1), \ 1 < k \leqslant m;$$

$$\omega(m+1, k, j) = k\omega(m, k, j) + (j+1)\omega(m, k, j+1) + \omega(m, k-1, j-1) + \omega(m, k-1, j), \ 1 \leqslant j < k \leqslant m. \tag{13}$$

Proof: If $m = 1$, then $k = 1$ and $j = 0$ or 1; in that case the relation (13) is trivial. Also, since

$$\omega(m, k, 0) = \binom{m}{k}k^{m-k}$$

and

$$\omega(m, k, 1) = (k^{m-k+1} - (k-1)^{m-k+1})\binom{m}{k-1},$$

we have immediately

$$k\omega(m, k, 0) + \omega(m, k-1, 0) + \omega(m, k, 1) = \omega(m+1, k, 0), \ 1 < k \leqslant m.$$

Now, for $1 \leqslant j < k$,

$$j![k\omega(m, k, j) + (j+1)\omega(m, k, j+1) + \omega(m, k-1, j-1) + \omega(m, k-1, j)]$$
$$= k\binom{m}{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m-k+j} + \binom{m}{k-j-1}\sum_{s=0}^{j+1}(-1)^s\binom{j+1}{s}(k-s)^{m-k+j+1}$$
$$+ j\binom{m}{k-j}\sum_{s=0}^{j-1}(-1)^s\binom{j-1}{s}(k-1-s)^{m-k+j} + \binom{m}{k-j-1}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-1-s)^{m-k+j+1}$$
$$= \binom{m}{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m+1-k+j} + \binom{m}{k-j-1}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m+1-k+j}$$
$$= \binom{m+1}{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}(k-s)^{m+1-k+j} = j!\omega(m+1, k, j).$$

This completes the proof of Lemma 2.

3. PROOFS OF THE THEOREMS

The proof of Theorem 2 is similar to that of Theorem 3; it suffices to define the sequence corresponding to (11*) below in an appropriate manner.

Proof of Theorem 1: Let us verify that if $G(z) := f(z^z)$ then

$$G_m^{(2)}(z) = \sum_{k=1}^{m}\sum_{j=0}^{k}\omega(m, k, j)z^k(\ln z)^j f_k^{(2)}(z^z). \tag{14}$$

It is sufficient to show that if we write

$$G_m^{(2)}(z) = \sum_{k=1}^{m}\sum_{j=0}^{k}w(m, k, j)z^k(\ln z)^j f_k^{(2)}(z^z)$$

then the sequence $w(m, k, j)$ satisfies the same recurrence relation (13) as $\omega(m, k, j)$ with the same initial conditions. Observe that

$$(f+g)_1^{(2)}(z) \equiv f_1^{(2)}(z) + g_1^{(2)}(z);$$

it follows from (7′) that

$$\begin{aligned} G_{m+1}^{(2)}(z) &= \sum_{k=1}^{m}\sum_{j=0}^{k} kw(m, k, j)z^k(\ln z)^j f_k^{(2)}(z^z) \\ &+ \sum_{k=1}^{m}\sum_{j=0}^{k} jw(m, k, j)z^k(\ln z)^{j-1} f_k^{(2)}(z^z) \\ &+ \sum_{k=1}^{m}\sum_{j=0}^{k} w(m, k, j)z^{k+1}(\ln z)^{j+1} f_{k+1}^{(2)}(z^z) \\ &+ \sum_{k=1}^{m}\sum_{j=0}^{k} w(m, k, j)z^{k+1}(\ln z)^{j} f_{k+1}^{(2)}(z^z). \end{aligned} \tag{15}$$

Relation (13) then follows immediately if we change, respectively, j to $j+1$, j to $j-1$ and k to $k-1$, and k to $k-1$ in the second, third, and fourth double summation of the right-hand side of (15). To see that $w(m, k, j)$ satisfies the same initial conditions as $\omega(m, k, j)$, we may use the observations made after the definition (3).

Now, using (12′) and (14), then (12), we obtain

$$\begin{aligned} G_m^{(1)}(z) &= \sum_{k=1}^{m} (-1)^{k+m} S_1(m, k)z^{-m} G_k^{(2)}(z) \\ &= \sum_{k=1}^{m}\sum_{\ell=1}^{k}\sum_{s=0}^{\ell} (-1)^{k+m} S_1(m, k)\omega(k, \ell, s)z^{\ell-m}(\ln z)^s \cdot f_\ell^{(2)}(z^z) \\ &= \sum_{k=1}^{m}\sum_{\ell=1}^{k}\sum_{s=0}^{\ell}\sum_{r=1}^{\ell} (-1)^{k+m} S_1(m, k)S(\ell, r)\omega(k, \ell, s)z^{rz+\ell-m}(\ln z)^s f_r^{(1)}(z^z). \end{aligned}$$

Proof of Theorem 3: It remains only to prove (11). That formula is clear for $m = 2$. Suppose that it is satisfied for a given $m > 2$. Then

$$\begin{aligned} (f^{-1})_{m+1}^{(2)}(z) &= \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \\ &\cdot \prod_{i=2}^{m} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i} \\ &\cdot \sum_{j=2}^{m} k_j \frac{\ln f_{j+1}^{(3)}(f^{-1}(z))}{\ln f_j^{(3)}(f^{-1}(z))} (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1} \\ &- \sum_{k=1}^{m-1} \sum_{\pi_1(m,k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1, \ldots, k_m) \\ &\cdot \prod_{i=2}^{m} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i} \cdot \ln f_2^{(3)}(f^{-1}(z)) \\ &\cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-2}. \end{aligned} \tag{16}$$

Let us put

$$k_i^{(1)} = \begin{cases} k_2 + 1, & i = 2 \\ k_i, & 2 < i \leqslant m \\ 0, & i = m + 1, \end{cases}$$

$$k_i^{(j)} = \begin{cases} k_i, & 2 \leqslant i < j \\ k_j - 1, & i = j \\ k_{j+1} + 1, & i = j + 1 \\ k_i, & j + 1 < i \leqslant m \\ 0, & i = m + 1,\ 2 \leqslant j < m, \end{cases} \tag{11*}$$

and

$$k_i^{(m)} = \begin{cases} k_i, & 2 \leqslant i < m \\ k_m - 1, & i = m \\ 1, & i = m + 1. \end{cases}$$

We have

$$\sum_{i=2}^{m+1} ik_i^{(1)} = m + k + 1, \quad \sum_{i=2}^{m+1} k_i^{(1)} = k + 1,$$

and

$$\sum_{i=2}^{m+1} ik_i^{(j)} = m + k, \quad \sum_{i=2}^{m+1} k_i^{(j)} = k, \quad 1 < j \leqslant m.$$

Identity (16) may thus be written in the form

$$\begin{aligned}(f^{-1})^{(2)}_{m+1}(z) = &\sum_{j=2}^{m} \sum_{k=1}^{m-1} \sum_{\pi_1^{(j)}(m+1,\,k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1^{(j)}, \ldots, k_m^{(j)})(j+1)k_{j+1}^{(j)} \\ &\cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^{(j)}} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1} \\ &- \sum_{k=1}^{m-1} \sum_{\pi_1^{(1)}(m+1,\,k+1)} (-1)^k \frac{(m+k)!}{m!} c_1(k_1^{(1)}, \ldots, k_m^{(1)}) \cdot 2k_2^{(1)} \\ &\cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^{(1)}} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-2},\end{aligned} \tag{17}$$

where $\pi_1^{(j)}(m + 1, k)$ means that the summation is extended over the numbers $k_2^{(j)}$, $\ldots$, $k_m^{(j)}$, related to the numbers $k_2, \ldots, k_m$ by (11*), satisfying

$$2k_2^{(j)} + \cdots + mk_m^{(j)} = m + k, \quad k_2^{(j)} + \cdots + k_m^{(j)} = k, \quad 1 < j \leqslant m;$$

$\pi_1^{(1)}(m + 1, k + 1)$ means that

$$2k_2^{(1)} + \cdots + mk_m^{(1)} = m + k + 1, \quad k_2^{(1)} + \cdots + k_m^{(1)} = k + 1.$$

We have put

$$c_1(k_1^{(j)}, \ldots, k^{(j)}) := \frac{m!}{k_2^{(j)}! \ldots k_m^{(j)}!(2!)^{k_2^{(j)}} \ldots (m!)^{k_m^{(j)}}}, \quad 1 \leqslant j \leqslant m.$$

Replacing k by $k - 1$ in the last summation of (17), we readily obtain

$$(f^{-1})^{(2)}_{m+1}(z) = \sum_{j=2}^{m} \sum_{k=1}^{m-1} \sum_{\pi_1^{(j)}(m+1,k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1^{(j)}, \ldots, k_m^{(j)})(j+1)k_{j+1}^{(j)} \cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^{(j)}} \cdot \ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1} + \sum_{k=2}^{m} \sum_{\pi_1^{(1)}(m+1,k)} (-1)^k \frac{(m+k-1)!}{m!} c_1(k_1^{(1)}, \ldots, k_m^{(1)}) \cdot 2k_2^{(1)} \cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^{(1)}} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1}. \tag{18}$$

Now, let $(k_2^*, \ldots, k_{m+1}^*)$ be a solution of the system

$$2k_2^* + \cdots + (m+1)k_{m+1}^* = m + k,$$

$$k_2^* + \cdots + k_{m+1}^* = k,$$

$$k_j^* \geqslant 0,\ 1 < j \leqslant m+1,\ (1 \leqslant k \leqslant m).$$

(i) If $k_2^* \neq 0$, then $k_{m+1}^* = 0$ (otherwise, $k_{m+1}^* = 1$ and $2K_2^* + \cdots + mk_m^* = k - 1 = k_2^* + \cdots + k_m^*$, which implies that $k_2^* = \cdots = k_m^* = 0$); in that case, to each solution $(k_2^*, \ldots, k_m^*, 0)$ there corresponds a solution $(k_2^{(1)}, \ldots, k_m^{(1)}, 0)$; it is possible, since the hypothesis $k_2^* \neq 0$ implies that $k_2 = k_2^{(1)} - 1 = k_2^* - 1 \geqslant 0$. Conversely, to each solution $(k_2^{(1)}, \ldots, k_{m+1}^{(1)})$, there corresponds a solution $(k_2^*, \ldots, k_m^*, k_{m+1}^* = 0)$.

(ii) Suppose that $1 < j < m$. If $k_{j+1}^* \neq 0$ then $k_{m+1}^* = 0$; in that case, to each solution $(k_2^*, \ldots, k_{m+1}^*)$, there corresponds a solution $(k_2^{(j)}, \ldots, k_{m+1}^{(j)} = 0)$; it is possible, since $k_{j+1} = k_{j+1}^{(j)} - 1 = k_{j+1}^* - 1 \geqslant 0$.

(iii) If $k_{m+1}^* \neq 0$, then $k_{m+1}^* = 1$ and $k_2^* = \cdots = k_m^* = 0$, $k = 1$. In that case, to the solution $(0, \ldots, 0, k_{m+1}^* = 1)$, there corresponds the solution $(0, \ldots, 0, k_{m+1}^{(m)} = 1)$.

Rearranging the terms in the summations of (18), we may thus write

$$(f^{-1})^{(2)}_{m+1}(z) = \sum_{j=2}^{m} \sum_{k=1}^{m-1} \sum_{\pi_1^*(m+1,k)} (-1)^k \frac{(m+k-1)!}{(m+1)!} c_1(k_1^*, \ldots, k_{m+1}^*)(j+1)k_{j+1}^* \cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^*} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1} + \sum_{k=2}^{m} \sum_{\pi_1^*(m+1,k)} (-1)^k \frac{(m+k-1)!}{(m+1)!} c_1(k_1^*, \ldots, k_{m+1}^*) \cdot 2k_2^* \cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^*} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-k-1}, \tag{19}$$

where

$$2k_2^* + \cdots + (m+1)k_{m+1}^* = m + k, \; k_2^* + \cdots + k_{m+1}^* = k,$$

and

$$c_1(k_1^*, \ldots, k_{m+1}^*) := \frac{(m+1)!}{k_1^* \ldots k_{m+1}^*!(1!)^{k_1^*} \ldots ((m+1)!)^{k_{m+1}^*}}.$$

In the first summation of (19) we may add the terms corresponding to $k = m$ since $2k_2^* + \cdots + (m+1)k_{m+1}^* = 2m$, $k_2^* + \cdots + k_{m+1}^* = m$ imply

$$(m-1)k_{m+1}^* + \cdots + 2k_4^* + k_3^* = 0,$$

i.e., $k_3^* = \cdots = k_{m+1}^* = 0$. Similarly, we may add, in the second summation of (19), the terms corresponding to $k = 1$. Writing

$$\sum_{j=2}^{m} (j+1)k_{j+1}^* = m + k - 2k_2^*,$$

we obtain

$$(f^{-1})_{m+1}^{(2)} = \sum_{k=1}^{m} \sum_{\pi_1^*(m+1,k)} (-1)^k \frac{(m+k)!}{(m+1)!} c_1(k_1^*, \ldots, k_{m+1}^*) \cdot \prod_{i=2}^{m+1} (\ln f_i^{(3)}(f^{-1}(z)))^{k_i^*} \cdot (\ln f_1^{(3)}(f^{-1}(z)))^{-m-1-k}. \tag{20}$$

This completes the proof of Theorem 3.

4. SOME REMARKS AND EXAMPLES

4.1 Remark on Taylor's formula: Let us write

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!}(g(z - z_0))^k, \; a_0 := f(z_0). \tag{21}$$

We have, in a neighborhood of $z = z_0$, $(g(0) = 0)$,

$$a_k = (f(z_0 + g^{-1}(z))^{(k)}(z = 0).$$

Put

$$f_1(z_0) := a_1 = \frac{f'(z_0 + g^{-1}(0))}{g'(g^{-1}(0))} \text{ and } f_k := (f_{k-1})_1, \; k > 1. \tag{22}$$

In order that $a_k \equiv f_k(z_0)$, we must have

$$(f(z_0 + g^{-1}(z)))^{(k)}(z = 0) \equiv \frac{f^{(k)}(z_0 + g^{-1}(0))}{(g'(g^{-1}(0))^k},$$

whence

$$f(z_0 + g^{-1}(z)) \equiv \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0 + g^{-1}(0))}{k!}\left(\frac{z}{g'(g^{-1}(0))}\right)^k = f\left(\frac{z}{g'(g^{-1}(0))} + z_0 + g^{-1}(0)\right),$$

in a neighborhood of $z = 0$. It follows that if g is normalized by the conditions

$$g(0) = 0,\ g'(0) = 1 \tag{24}$$

then $g(z) \equiv z$. The unique function g, normalized by (24), for which the expansion (21) is valid, where a_k is the k^{th} iteration of the operator induced by $f_1 := a_1$, is the identity function $g(z) = z$; in that case, $f_1 = f'$. A similar argument may be made for expansions of the form

$$\sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\ln \frac{z}{z_0}\right)^k,\quad \sum_{k=0}^{\infty} \frac{\ln a_k}{k!}(z - z_0)^k,\quad \sum_{k=0}^{\infty} \frac{\ln a_k}{k!}\left(\ln \frac{z}{z_0}\right)^k. \tag{25}$$

It is in fact easy to come down to the previous case. For the expansions (25) we have, respectively, $f_1 = f_1^{(2)}$, $f_1 = f_1^{(3)}$, $f_1 = f_1^{(4)}$ [see (6), (7), and (7′)].

It is of interest to observe here that for expansions of the form

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!}(g(z) - g(z_0))^k,\quad a_0 := f(z_0), \tag{21′}$$

we have always that a_k is the k^{th} iteration of the operator induced by

$$f_1(z_0) := \frac{f'(z_0)}{g'(z_0)}.$$

To see this, we may easily show that

$$f_k(z_0) = \left.\frac{\partial^k f(g^{-1}(z + g(z_0)))}{\partial z^k}\right|_{z=0},\quad k = 1, 2, 3, \ldots .$$

4.2 (i) Let us take $f(z) = e^z$, then $z = 1$, in (4); we obtain:

$$(\exp(z^z))_m^{(1)}(z = 1) = e\sum_{k=1}^{m}\sum_{\ell=1}^{k}\sum_{r=1}^{\ell}(-1)^{k+m}S_1(m, k)S(\ell, r)\cdot\binom{k}{\ell}\ell^{k-\ell}. \tag{26}$$

(ii) If $g(z) = z^z$ in (9′), then we obtain, using (14) and $g_j^{(4)}(z) = z^z e^{jz}$, $j = 0, 1, 2, \ldots$, the identity

$$\sum_{\pi(m,k)} c(k_1, \ldots, k_m)\prod_{j=1}^{m}(z + j)^{k_j} = \sum_{j=0}^{k}\omega(m, k, j)z^j,\ z \in \mathbb{C}. \tag{27}$$

Note that we can deduce from (8) (see [5], p. 191) the relation

$$\sum_{\pi(m,k)} \frac{k!}{k_1! \ldots k_m!}\prod_{j=1}^{m} j^{k_j} = \binom{m + k - 1}{m - k},\ 1 \leqslant k \leqslant m.$$

(iii) Lagrange expansion [concerning a root of equations of the form $z = a + \xi\phi(z)$, $\xi \to 0$] in conjunction with (8) may be used to prove the formula

$$\sum_{\pi(m,k)} c(k_1, \ldots, k_m)\prod_{j=1}^{m}((\phi^j(a))^{(j-1)})^{k_j} \equiv \binom{m-1}{k-1}(\phi^m(a))^{(m-k)}, \tag{28}$$

$$1 \leqslant k \leqslant m,$$

which implies that

$$\sum_{\pi(m)} c(k_1, \ldots, k_m)\prod_{j=1}^{m}((\phi^j(a))^{(j-1)})^{k_j} \equiv e^{-a}(\phi^m(a)e^a)^{(m-1)}, \tag{29}$$

where $\pi(m)$ means that the summation is extended over all nonnegative integers $k_1, \ldots, k_m$ such that $k_1 + 2k_2 + \cdots + mk_m = m$.

REFERENCES

1. L. Carlitz. "Weighted Stirling Numbers of the First and Second Kind, I," and "Weighted Stirling Numbers of the First and Second Kind, II." *The Fibonacci Quarterly 18*, no. 2 (1980):147-62, and no. 3 (1980):242-57.
2. Ch. A. Charalambides. "On Weighted Stirling and Other Related Numbers and Some Combinatorial Applications." *The Fibonacci Quarterly 22*, no. 4 (1984): 296-309.
3. L. Comtet. *Analyse Combinatoire*, vols. I and II. Paris: Presses Universitaires de France, 1970.
4. F. T. Howard. "Weighted Associated Stirling Numbers." *The Fibonacci Quarterly 22*, no. 2 (1984):156-65.
5. J. Riordan. *Combinatorial Identities*. New York: Wiley & Sons, 1968.
6. St. Ruscheweyh. *Convolutions in Geometric Function Theory*. Séminaire de Mathématiques Supérieurs. Montréal: Les Presses de l'Université de Montréal, 1982.
7. I. J. Schwatt. *An Introduction to the Operations with Series*. New York: Chelsea Publishing Company, 1924.
8. R. Steven. "The Formula of Faa Di Bruno." *Amer. Math. Monthly 87*, no. 10 (1980):805-09.

◆◇◆◇◆