EXPONENTIAL GENERATING FUNCTIONS FOR PELL POLYNOMIALS

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1. INTRODUCTION

Following our description [6] of the properties of the ordinary generating functions of Pell polynomials $P_n(x)$ and Pell-Lucas polynomials $Q_n(x)$ [3], we offer here a compact exposition of similar properties of the exponential generating functions of these polynomials.

Earlier authors have written about the exponential generating functions of the Fibonacci numbers [2] and of generalized Fibonacci numbers [7].

Details of the main properties of the Pell-type polynomials may be found in [3] and [4], and will be assumed, where necessary. For visual simplicity, we will abbreviate the functional notation thus: $P_n(x) \equiv P_n$, $Q_n(x) \equiv Q_n$.

Binet forms of P_n and Q_n are

$$P_n = (\alpha^n - \beta^n)/(\alpha - \beta) \tag{1.1}$$

and

$$Q_n = \alpha^n + \beta^n, \tag{1.2}$$

where

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \tag{1.3}$$

$$(\text{so } \alpha + \beta = 2x,\ \alpha\beta = -1,\ \alpha - \beta = 2\sqrt{x^2 + 1})$$

are the roots of

$$\lambda^2 - 2x\lambda - 1 = 0. \tag{1.4}$$

Some symbolism we shall employ include:

$$\nabla = (1 - 2xz - z^2)^{-1} \quad (= \Delta \text{ in [6] with } y \text{ replaced by } z) \tag{1.5}$$

$$\nabla_{(m)} = (1 - Q_m z + (-1)^m z^2)^{-1}, \text{ i.e., } \nabla_{(1)} \equiv \nabla \tag{1.6}$$

$$\nabla' = (1 + 2xz - z^2)^{-1}, \text{ i.e., replace } z \text{ by } -z \text{ in (1.5)} \tag{1.7}$$

$$\nabla^{(2)} \equiv \Delta^{(2)} \text{ in [6] with } y \text{ replaced by } z \tag{1.8}$$

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \tag{1.9}$$

$$P^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} \tag{1.10}$$

Usage of the matrix P (1.9) is to be found, for example, in [3], [4], [5], and [6]. Inevitably, some of the simpler results for Pell-type polynomials in the ensuing pages may have been obtained by other methods in our papers listed as references.

2. BASIC MATERIAL

Write

$$P(x,\ y,\ 0) = \frac{e^{\alpha y} - e^{\beta y}}{\alpha - \beta} = \sum_{r=0}^{\infty} \frac{P_r y^r}{r!} \tag{2.1}$$

and

$$Q(x,\ y,\ 0) = e^{\alpha y} + e^{\beta y} = \sum_{r=0}^{\infty} \frac{Q_r y^r}{r!}. \tag{2.2}$$

Both (2.1) and (2.2) satisfy

$$\frac{\partial^2 t}{\partial y^2} - 2x\frac{\partial t}{\partial y} - t = 0. \tag{2.3}$$

From (2.1)

$$P(x,\ y,\ k) = \frac{\partial^k}{\partial y^k} P(x,\ y,\ 0) = \sum_{r=0}^{\infty} \frac{P_{r+k} y^r}{r!}, \tag{2.4}$$

whence

$$P(x,\ y,\ n+1) - 2xP(x,\ y,\ n) - P(x,\ y,\ n-1) = 0. \tag{2.5}$$

Also

$$Q(x,\ y,\ k) = \frac{\partial^k}{\partial y^k} Q(x,\ y,\ 0) = \sum_{r=0}^{\infty} \frac{Q_{r+k} y^r}{r!}, \tag{2.6}$$

whence

$$Q(x,\ y,\ n+1) - 2xQ(x,\ y,\ n) - Q(x,\ y,\ n-1) = 0. \tag{2.7}$$

Formulas (2.5) and (2.7) suggest the matrix representations:

$$\begin{bmatrix} P(x,\ y,\ n) \\ P(x,\ y,\ n-1) \end{bmatrix} = P^{n-1} \begin{bmatrix} P(x,\ y,\ 1) \\ P(x,\ y,\ 0) \end{bmatrix} \tag{2.8}$$

$$\begin{bmatrix} Q(x,\ y,\ n) \\ Q(x,\ y,\ n-1) \end{bmatrix} = P^{n-1} \begin{bmatrix} Q(x,\ y,\ 1) \\ Q(x,\ y,\ 0) \end{bmatrix} \tag{2.9}$$

$$P(x,\ y,\ n) = [1 \quad 0]P^{n-1} \begin{bmatrix} P(x,\ y,\ 1) \\ P(x,\ y,\ 0) \end{bmatrix} \tag{2.10}$$

$$Q(x,\ y,\ n) = [1 \quad 0]P^{n-1} \begin{bmatrix} Q(x,\ y,\ 1) \\ Q(x,\ y,\ 0) \end{bmatrix} \tag{2.11}$$

3. PROPERTIES OF EXPONENTIAL GENERATING FUNCTIONS

First, from (2.4) and (2.1) or by matrices,

$$\begin{aligned} P(x,\ y,\ n+1) + P(x,\ y,\ n-1) &= \frac{\alpha^{n+1}e^{\alpha y} - \beta^{n+1}e^{\beta y} + \alpha^{n-1}e^{\alpha y} - \beta^{n-1}e^{\beta y}}{\alpha - \beta} \\ &= \alpha^n e^{\alpha y} + \beta^n e^{\beta y} \qquad (3.1) \\ &= Q(x,\ y,\ n) \quad \text{by (2.6)} \end{aligned}$$

while, similarly,

$$Q(x, y, n+1) + Q(x, y, n-1) = 4(x^2+1)P(x, y, n). \tag{3.2}$$

Generalizations, with variations, of (3.1) and (3.2) are:

$$P(x, y, n+r) + (-1)^r P(x, y, n-r) = Q_r P(x, y, n) \tag{3.3}$$

$$P(x, y, n+r) - (-1)^r P(x, y, n-r) = P_r Q(x, y, n) \tag{3.4}$$

$$Q(x, y, n+r) + (-1)^r Q(x, y, n-r) = Q_r Q(x, y, n) \tag{3.5}$$

$$Q(x, y, n+r) - (-1)^r Q(x, y, n-r) = 4(x^2+1)P_r P(x, y, n) \tag{3.6}$$

An elementary property is, by (2.1), (2.6), and (2.4),

$$P(x, y, n)Q(x, y, n) = P(x, 2y, 2n)/2^n. \tag{3.7}$$

Combining (3.3) and (3.4) with (3.7), we arrive at:

$$P^2(x, y, n+r) - P^2(x, y, n-r) = P_{2r}P(x, 2y, 2n)/2^n \tag{3.8}$$

$$Q^2(x, y, n+r) - Q^2(x, y, n-r) = 4(x^2+1)P_{2r}P(x, 2y, 2n)/2^n \tag{3.9}$$

For variety, we use matrices to demonstrate the *Simson formula* (3.10) for $P(x, y, n)$. Details are:

$$P(x, y, n+1)P(x, y, n-1) - P^2(x, y, n) \tag{3.10}$$

$$= \begin{vmatrix} P(x, y, n+1) & P(x, y, n) \\ P(x, y, n) & P(x, y, n-1) \end{vmatrix}$$

$$= \left| P^n \begin{bmatrix} P(x, y, 1) \\ P(x, y, 0) \end{bmatrix} \;\vdots\; P^{n-1} \begin{bmatrix} P(x, y, 1) \\ P(x, y, 0) \end{bmatrix} \right| \qquad \text{by (2.8)}$$

$$= (-1)^{n-1} \begin{vmatrix} P(x, y, 2) & P(x, y, 1) \\ P(x, y, 1) & P(x, y, 0) \end{vmatrix} \qquad \text{by (2.8)} \quad [\,|P^{n-1}| = (-1)^{n-1}]$$

$$= (-1)^{n-1}\{(\alpha^2 e^{\alpha y} - \beta^2 e^{\beta y})(e^{\alpha y} - e^{\beta y}) - (\alpha e^{\alpha y} - \beta e^{\beta y})^2\}/(\alpha - \beta)^2 \qquad \text{by (2.1) and (2.4)}$$

$$= (-1)^{n-1}\{-(\alpha^2 + \beta^2 - 2\alpha\beta)e^{(\alpha+\beta)y}\}/(\alpha - \beta)^2$$

$$= (-1)^n e^{2xy} \qquad \text{by (1.3)}$$

Likewise,

$$Q(x, y, n+1)Q(x, y, n-1) - Q^2(x, y, n)$$
$$= (-1)^{n-1}4(x^2+1)e^{2xy}. \tag{3.11}$$

The clear similarity of the results in this section with the corresponding formulas for P_n and Q_n is noticeable.

Obviously, the number of relationships involving exponential generating functions themselves alone is extensive. Three such are, for example:

$$P(x, y, n)P(x, y, r+1) + P(x, y, n-1)P(x, y, r)$$
$$= P(x, 2y, n+r)/2^{n+r}; \tag{3.12}$$

$$Q(x, y, n)Q(x, y, r+1) + Q(x, y, n-1)Q(x, y, r)$$
$$= 4(x^2+1)P(x, 2y, n+r)/2^{n+r}; \tag{3.13}$$

and

$$P(x, y, n)Q(x, y, r+1) + P(x, y, n-1)Q(x, y, r) = Q(x, 2y, n+r)/2^{n+r}. \tag{3.14}$$

Put $r = n - 1$ in (3.12) and (3.13) to get, in succession,

$$P^2(x, y, n) + P^2(x, y, n-1) = P(x, 2y, 2n-1)/2^{2n-1} \tag{3.15}$$

and

$$Q^2(x, y, n) + Q^2(x, y, n-1) = 4(x^2+1)P(x, 2y, 2n-1)/2^{2n-1}. \tag{3.16}$$

Finally,

$$P(x, y, m)Q(x, y, n) + P(x, y, n)Q(x, y, m) = P(x, 2y, m+n)/2^{m+n-1} \tag{3.17}$$

and

$$Q(x, y, m)Q(x, y, n) + 4(x^2+1)P(x, y, m)P(x, y, n) = Q(x, 2y, m+n)/2^{m+n-1} \tag{3.18}$$

Reverting now to the formulas relating exponential generating functions to Pell polynomials, we may establish, either by means of the definitions or by the matrix representations, the following:

$$P(x, y, n+r) = P_r P(x, y, n+1) + P_{r-1}P(x, y, n) \tag{3.19}$$

$$Q(x, y, n+r) = P_r Q(x, y, n+1) + P_{r-1}Q(x, y, n) = Q_r P(x, y, n+1) + Q_{r-1}P(x, y, n) \tag{3.20}$$

$$4(x^2+1)P(x, y, n+r) = Q_r Q(x, y, n+1) + Q_{r-1}Q(x, y, n) \tag{3.21}$$

Special cases of interest occur when $r = n$ in (3.19)-(3.21).
Also,

$$P(x, y, n+r) = \tfrac{1}{2}\{P_r Q(x, y, n) + Q_r P(x, y, n)\}, \tag{3.22}$$

$$Q(x, y, n+r) = \tfrac{1}{2}\{4(x^2+1)P_r P(x, y, n) + Q_r Q(x, y, n)\}, \tag{3.23}$$

$$P(x, y, n+r)P(x, y, n-r) - P^2(x, y, n) = (-1)^{n-r+1}P_r^2 e^{2xy}, \tag{3.24}$$

$$Q(x, y, n+r)Q(x, y, n-r) - Q^2(x, y, n) = (-1)^{n-r}4(x^2+1)P_r^2 e^{2xy}. \tag{3.25}$$

Results (3.24) and (3.25) are the *generalized Simson formulas*.
Lastly, in this section,

$$P(x, y, n)P(x, y, n+r+1) - P(x, y, n-s)P(x, y, n+r+s+1) = (-1)^{n-s}P_{r+s+1}P_s e^{2xy}, \tag{3.26}$$

and

$$Q(x, y, n)Q(x, y, n+r+1) - Q(x, y, n-s)Q(x, y, n+r+s+1) = (-1)^{n-s+1}4(x^2+1)P_{r+s+1}P_s e^{2xy}. \tag{3.27}$$

4. SERIES INVOLVING EXPONENTIAL GENERATING FUNCTIONS

Rearranging (2.5) and (2.7), and adding, we find

$$\sum_{r=1}^{n} P(x, y, r) = \{P(x, y, n+1) + P(x, y, n) - P(x, y, 1) - P(x, y, 0)\}/2x \tag{4.1}$$

and

$$\sum_{r=1}^{n} Q(x, y, r) = \{Q(x, y, n+1) + Q(x, y, n) - Q(x, y, 1) - Q(x, y, 0)\}/2x. \tag{4.2}$$

Binet forms give us the difference equations,

$$P(x, y, m(r+1)+k) - Q_m P(x, y, mr+k) + (-1)^m P(x, y, m(r-1)+k) = 0 \tag{4.3}$$

and

$$Q(x, y, m(r+1)+k) - Q_m Q(x, y, mr+k) + (-1)^m Q(x, y, m(r-1)+k) = 0. \tag{4.4}$$

Using (4.3) and (4.4), we may derive

$$\sum_{r=1}^{n} P(x, y, mr+k) \tag{4.5}$$

$$= \frac{P(x, y, m(n+1)+k) - P(x, y, m+k) - (-1)^m\{P(x, y, mn+k) - P(x, y, k)\}}{Q_m - 1 - (-1)^m}$$

and

$$\sum_{r=1}^{n} Q(x, y, mr+k) \tag{4.6}$$

$$= \frac{Q(x, y, m(n+1)+k) - Q(x, y, m+k) - (-1)^m\{Q(x, y, mn+k) - Q(x, y, k)\}}{Q_m - 1 - (-1)^m}.$$

Next, (2.8) and (3.19) used in conjunction with the matrix property

$$P^2 = 2xP + I$$

yield

$$P^{2n}\begin{bmatrix} P(x, y, 1) \\ P(x, y, 0) \end{bmatrix} = (2xP + I)^n \begin{bmatrix} P(x, y, 1) \\ P(x, y, 0) \end{bmatrix}. \tag{4.7}$$

Equating corresponding elements, we obtain

$$P(x, y, 2n) = \sum_{r=0}^{n} \binom{n}{r}(2x)^r P(x, y, r) \tag{4.8}$$

and

$$P(x, y, 2n+1) = \sum_{r=0}^{n} \binom{n}{r}(2x)^r P(x, y, r+1). \tag{4.9}$$

Similarly,

$$Q(x, y, 2n) = \sum_{r=0}^{n} \binom{n}{r}(2x)^r Q(x, y, r) \tag{4.10}$$

and

$$Q(x, y, 2n+1) = \sum_{r=0}^{n} \binom{n}{r}(2x)^r Q(x, y, r+1). \tag{4.11}$$

Extensions of (4.10) and (4.11) to $P(x, y, 2n+j)$ and $Q(x, y, 2n+j)$ readily follow.

Now let us consider a variation of the type of sequence being summed. Applying the Simson formula (3.10), simplifying, and summing, we derive

$$\sum_{r=1}^{n} \frac{(-1)^{r-1}}{P(x, y, r)P(x, y, r+1)} = \frac{1}{e^{2xy}}\left\{\frac{P(x, y, n)}{P(x, y, n+1)} - \frac{P(x, y, 0)}{P(x, y, 1)}\right\}. \tag{4.12}$$

Similarly,

$$\sum_{r=1}^{n} \frac{(-1)^r}{Q(x, y, r)Q(x, y, r+1)}$$

$$= \frac{1}{e^{2xy}} \left\{ \frac{Q(x, y, n)}{Q(x, y, n+1)} - \frac{Q(x, y, 0)}{Q(x, y, 1)} \right\} \frac{1}{4(x^2+1)}. \tag{4.13}$$

5. ORDINARY GENERATING FUNCTIONS OF EXPONENTIAL GENERATING FUNCTIONS

Summing and using (2.5),

$$\sum_{r=0}^{\infty} P(x, y, r)z^r = (P(x, y, 0) + P(x, y, -1)z)\nabla \tag{5.1}$$

where $P(x, y, -1)$ is the *primitive function* of $P(x, y, 0)$ w.r.t. y.
Similarly,

$$\sum_{r=0}^{\infty} Q(x, y, r)z^r = (Q(x, y, 0) + Q(x, y, -1)z)\nabla, \tag{5.2}$$

$$\sum_{r=0}^{\infty} (-1)\ P(x, y, r)z^r = (P(x, y, 0) - P(x, y, -1)z)\nabla', \tag{5.3}$$

and

$$\sum_{r=0}^{\infty} (-1)^r Q(x, y, r)z^r = (Q(x, y, 0) - Q(x, y, -1)z)\nabla'. \tag{5.4}$$

More generally,

$$\sum_{r=0}^{\infty} P(x, y, mr+k)z^r = \{P(x, y, k) - (-1)^m P(x, y, -m+k)z\}\nabla_{(m)}, \tag{5.5}$$

and

$$\sum_{r=0}^{\infty} Q(x, y, mr+k)z^r = \{Q(x, y, k) - (-1)^m Q(x, y, -m+k)z\}\nabla_{(m)}. \tag{5.6}$$

Induction gives

$$\frac{\partial^n}{\partial z^n} \sum_{r=0}^{\infty} P(x, y, r)z^r = n! \left\{ \sum_{r=0}^{n+1} \binom{n+1}{r} P(x, y, n-r)z^r \right\} \nabla^{n+1} \tag{5.7}$$

and

$$\frac{\partial^n}{\partial z^n} \sum_{r=0}^{\infty} Q(x, y, r)z^r = n! \left\{ \sum_{r=0}^{n+1} \binom{n+1}{r} Q(x, y, n-r)z^r \right\} \nabla^{n+1} \tag{5.8}$$

with extensions when r is replaced by $r+m$.
Equating coefficients of z^r in (5.7) and (5.8) yields, in turn,

$$P(x, y, n+r) = \left\{ \sum_{i=0}^{n+1} \binom{n+1}{i} P(x, y, n-i) P_{r+1-i}^{(n)} \right\} \Big/ \binom{n+r}{r} \tag{5.9}$$

and

$$Q(x, y, n+r) = \left\{ \sum_{i=0}^{n+1} \binom{n+1}{i} Q(x, y, n-i) P_{r+1-i}^{(n)} \right\} \Big/ \binom{n+r}{r}, \tag{5.10}$$

since

$$\nabla^{n+1} = \sum_{t=0}^{\infty} P_{t+1}^{(n)} z^t,$$

where $\{P_i^{(n)}\}$, $i = 1, 2, 3, \ldots$ is the n^{th} *convolution sequence* for Pell polynomials [4].

Now, by (2.1) and (2.4), we can demonstrate that

$$P^2(x, y, r+1) - Q_2 P^2(x, y, r) + P^2(x, y, r-1) = 2(-1)^r e^{2xy}. \quad (5.11)$$

Using this as a difference equation, we obtain

$$\sum_{r=1}^{n} P^2(x, y, r) = [P^2(x, y, n+1) - P^2(x, y, 1) - \{P^2(x, y, n) - P^2(x, y, 0)\} + 2(1-(-1)^n)e^{2xy}]/4x^2 \quad (5.12)$$

and

$$\sum_{r=0}^{\infty} P^2(x, y, r)z^r = [P^2(x, y, 0) + z\{P^2(x, y, 0) - P^2(x, y, -1)\} - P^2(x, y, -1)z^2 + 2ze^{2xy}]\nabla^{(2)}/(1+z) \quad (5.13)$$

by (1.8).

Furthermore,

$$P^2(x, y, n+3) - (4x^2+1)P^2(x, y, n+2) - (4x^2+1)P^2(x, y, n+1) + P^2(x, y, n) = 0, \quad (5.14)$$

$$\sum_{r=0}^{\infty} \frac{P_{mr+k}y^r}{r!} = (\alpha^k e^{\alpha^m y} - \beta^k e^{\beta^m y})/(\alpha - \beta), \quad (5.15)$$

and

$$\sum_{r=0}^{\infty} \frac{P_r^2 y^r}{r!} = (e^{\alpha^2 y} + e^{\beta^2 y} - 2e^{-y})/(\alpha - \beta)^2. \quad (5.16)$$

6. FURTHER APPLICATIONS OF EXPONENTIAL GENERATING FUNCTIONS

Techniques employed for Fibonacci numbers in [1] are now cultivated for Pell polynomials.

To illustrate the method, we show that

$$P_{2n} = \sum_{r=0}^{n} \binom{n}{r}(2x)^r P_r. \quad (6.1)$$

Consider

$$\begin{aligned} A &= \{(e^{2\alpha xy} - e^{2\beta xy})e^y\}/(\alpha - \beta) \quad (6.2) \\ &= \{e^{(2\alpha x+1)y} - e^{(2\beta x+1)y}\}/(\alpha - \beta) \\ &= (e^{\alpha^2 y} - e^{\beta^2 y})/(\alpha - \beta) \qquad \text{by (1.3)} \\ &= \sum_{n=0}^{\infty} \frac{P_{2n}y^n}{n!} \qquad \text{by (1.1).} \end{aligned}$$

However, also,

$$\begin{aligned} A &= \left\{\sum_{n=0}^{\infty} \frac{(2x)^n P_n y^n}{n!}\right\}\left\{\sum_{n=0}^{\infty} \frac{y^n}{n!}\right\} \qquad \text{by (6.2) and (1.1)} \quad (6.3) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n} \frac{(2x)^i P_i}{i!(n-i)!}\right\}y^n. \end{aligned}$$

By equating the coefficients of y^n in (6.2) and (6.3), we get

$$\frac{P_{2n}}{n!} = \sum_{i=0}^{n} \frac{(2x)^i P_i}{i!(n-1)!}, \tag{6.4}$$

which is equivalent to (6.1).

Observe that (6.2) and (6.3) lead to

$$\frac{\partial^r A}{\partial y^r} = \sum_{n=0}^{\infty} \frac{P_{2n+2r} y^n}{n!} = \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^{n+r} \frac{(n+1)_r (2x)^i P_i y^n}{i!(n+r-i)!} \right\}$$

where $(n)_r$ is the rising factorial.

Hence,

$$P_{2(n+r)} = \sum_{i=0}^{n+r} \binom{n+r}{i} (2x)^i P_i, \tag{6.5}$$

which is an extension of (6.4).

Turning our attention to

$$B = (e^{\alpha y} - e^{\beta y}) e^{-2xy} / (\alpha - \beta), \tag{6.6}$$

we obtain, in a similar manner,

$$(-1)^{n+1} P_n = \sum_{i=0}^{n} \binom{n}{i} (-2x)^{n-i} P_i. \tag{6.7}$$

Likewise, from

$$C = (e^{\alpha^2 y} - e^{\beta^2 y}) e^{-y} / (\alpha - \beta), \tag{6.8}$$

we derive

$$(2x)^n P_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} P_{2i}. \tag{6.9}$$

Next, consider

$$\begin{aligned} D &= (e^{\alpha^m y} - e^{\beta^m y})(e^{\alpha^m y} + e^{\beta^m y}) / (\alpha - \beta) \qquad (6.10) \\ &= (e^{2\alpha^m y} - e^{2\beta^m y}) / (\alpha - \beta) \\ &= \sum_{n=0}^{\infty} \frac{2^n P_{mn} y^n}{n!} \quad \text{by (1.1).} \end{aligned}$$

Now, also,

$$D = \sum_{n=0}^{\infty} \left\{ \frac{P_{mn} y^n}{n!} \right\} \left\{ \sum_{n=0}^{\infty} \frac{Q_{mn} y^n}{n!} \right\} = \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^{n} \frac{P_{mi} Q_{m(n-i)}}{i!(n-i)!} \right\} y^n. \tag{6.11}$$

So

$$2^n P_{mn} = \sum_{i=0}^{n} \binom{n}{i} P_{mi} Q_{m(n-i)}. \tag{6.12}$$

If we investigate

$$E = (e^{\alpha^m y} - e^{\beta^m y})(e^{\alpha^m y} - e^{\beta^m y}) / (\alpha - \beta)^2, \tag{6.13}$$

we are led by the above process, eventually, to

$$2^n Q_{mn} - 2Q_m^n = 4(x^2 + 1)\sum_{r=0}^{n}\binom{n}{r}P_{mr}P_{m(n-r)}. \tag{6.14}$$

Similarly,

$$2\, Q_{mn} + 2Q_m^n = \sum_{r=0}^{n}\binom{n}{r}Q_{mr}Q_{m(n-r)}. \tag{6.15}$$

Suppose now that

$$\begin{aligned} F &= \{(e^{\alpha^{4m}y} - e^{\beta^{4m}y})e^y\}/(\alpha - \beta) \qquad (6.16)\\ &= \{e^{(\alpha^{4m}+1)y} - e^{(\beta^{4m}+1)y}\}/(\alpha - \beta)\\ &= \{e^{(\alpha^{4m}+\alpha^{2m}\beta^{2m})y} - e^{(\beta^{4m}+\alpha^{2m}\beta^{2m})y}\}/(\alpha - \beta)\\ &= \{e^{\alpha^{2m}(\alpha^{2m}+\beta^{2m})y} - e^{\beta^{2m}(\alpha^{2m}+\beta^{2m})y}\}/(\alpha - \beta)\\ &= \sum_{n=0}^{\infty}\frac{P_{2mn}Q_{2m}^n y^n}{n!} \quad \text{by (1.1) and (1.2).} \end{aligned}$$

But, also,

$$\begin{aligned} F &= \left\{\sum_{n=0}^{\infty}\frac{P_{4mn}y^n}{n!}\right\}\left\{\sum_{n=0}^{\infty}\frac{y^n}{n!}\right\} \quad \text{by (6.16) and (1.1)} \qquad (6.17)\\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\frac{P_{4mi}}{i!(n-i)!}\right\}y^n. \end{aligned}$$

Consequently,

$$P_{2mn}Q_{2m}^n = \sum_{i=0}^{n}\binom{n}{i}P_{4mi}. \tag{6.18}$$

Differentiating r times partially w.r.t. y the two expressions (6.16) and (6.17) for F, as we did earlier for A [cf. (6.5)], we obtain the extension of (6.18), namely,

$$P_{2m(n+r)}Q_{2m}^{n+r} = \sum_{i=0}^{n+r}\binom{n+r}{i}P_{4mi}. \tag{6.19}$$

Finally, consider

$$\begin{aligned} G &= (e^{\alpha^m y} - e^{\beta^m y})/(\alpha - \beta) \qquad (6.20)\\ &= \left\{e^{(\alpha P_m + P_{m-1})y} - e^{(\beta P_m + P_{m-1})y}\right\}/(\alpha - \beta)\\ &= \left\{e^{P_{m-1}y}(e^{\alpha P_m y} - e^{\beta P_m y})\right\}/(\alpha - \beta)\\ &= \left\{\sum_{n=0}^{\infty}\frac{P_{m-1}^n y^n}{n!}\right\}\left\{\sum_{n=0}^{\infty}\frac{P_n P_m^n y^n}{n!}\right\} \quad \text{by (1.1)}\\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\frac{P_{m-i}^i P_{n-i}P_m^{n-i}}{i!(n-i)!}\right\}y^n. \end{aligned}$$

Also,

$$G = \sum_{n=0}^{\infty}\frac{P_{mn}y^n}{n!} \quad \text{by (6.20) and (1.1).} \tag{6.21}$$

Then

$$P_{mn} = \sum_{i=0}^{n} \binom{n}{i} P_{m-1}^{i} P_{n-i} P_{m}^{n-i} = \sum_{i=0}^{n} \binom{n}{i} P_{m-1}^{n-i} P_i P_m^{i}, \tag{6.22}$$

whence

$$\frac{\partial^r G}{\partial y^r} = \sum_{n=0}^{\infty} \frac{P_{m(n+r)} y^n}{n!} = \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^{n+r} \frac{(n+1)_r P_m^i P_{m-1}^{n+r-i} P_i}{i!(n+r-i)!} \right\} y^n \tag{6.23}$$

and

$$P_{m(n+r)} = \sum_{i=0}^{n+r} \binom{n+r}{i} P_m^i P_{m-1}^{n+r-i} P_i. \tag{6.24}$$

The presentation in this article of the properties of the exponential generating functions of Pell and Pell-Lucas polynomials suffices to give us something of their mathematical flavor.

Important special cases of the Pell polynomials and Pell-Lucas polynomials are noted in [3] and may, for variety and visual convenience, be tabulated as:

	P_n	Q_n
$x = 1$	Pell numbers	Pell-Lucas numbers
$x = \frac{1}{2}$	Fibonacci numbers	Lucas numbers
$x \to \frac{1}{2}x$	Fibonacci polynomials	Lucas polynomials

Results given in this paper for exponential generating functions, and in [6] for ordinary generating functions, of P_n and Q_n may clearly be specialized to corresponding results for the tabulated mathematical entities.

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