POWERFUL k-SMITH NUMBERS

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(Submitted August 1985)

1. INTRODUCTION

Let $S(m)$ denote the sum of the digits of the positive integer $m > 1$, and $S_p(m)$ denote the sum of all the digits of all the prime factors of m. If k is a positive integer such that $S_p(m) = kS(m)$, m is called a k-Smith number, and when $k = 1$, simply, a Smith number [5].

A powerful number is an integer m with the property that if $p|m$ then $p^2|m$. The number of positive powerful numbers less than $x > 0$ is between $cx^{1/2} - 3x^{1/3}$ and $cx^{1/2}$, where $c \approx 2.173$ (see [1]). By actual count, for example, there are 997 powerful numbers less than 250,000.

Precious little is known about the frequency of occurrence of Smith numbers or of their distribution. Wilansky [5] has found 360 Smith numbers among the integers less than 10,000, and we have shown [2] that infinitely many k-Smith numbers exist ($k \geqslant 1$). In this paper, we investigate the existence of k-Smith numbers in two complementary sets: the set of powerful numbers and its complement. A basic relationship between $S_p(m)$ and the number $N(m)$ of digits of m is first obtained. We then show (not surprisingly) that there exist infinitely many k-Smith numbers ($k \geqslant 1$) which are not powerful numbers. Finally, we use the basic relationship to show that there exist infinitely many k-Smith numbers ($k > 1$) among the integers in each of the two categories of powerful numbers: square and nonsquare.

2. TWO LEMMAS

Lemma 1: If b, k, and n are positive integers, $k < n$, and

$$t = a_k 10^k + \cdots + a_1 10 + a_0$$

is an integer with $0 < a_0 \leqslant 5$ and $0 \leqslant a_i < 5$ for $1 \leqslant i \leqslant k$, then

$$S(t(10^n - 1)^2 \cdot 10^b) = 9n.$$

Proof: If in the product of t and $10^{2n} - 2 \cdot 10^n + 1$ we replace $a_0 10^{2n}$ by

$$(a_0 - 1)10^{2n} + 9 \cdot 10^{2n-1} + \cdots + 9 \cdot 10^{n+1} + 10 \cdot 10^n,$$

we obtain

$$\begin{aligned} t(10^n - 1)^2 \cdot 10^b = [&a_k 10^{2n+k} + \cdots + a_1 10^{2n+1} + (a_0 - 1)10^{2n} \\ &+ 9 \cdot 10^{2n-1} + \cdots + 9 \cdot 10^{n+k+1} \\ &+ (9 - 2a_k)10^{n+k} + \cdots + (9 - 2a_1)10^{n+1} \\ &+ (10 - 2a_0)10^n + a_k 10^k + \cdots + a_0] \cdot 10^b. \end{aligned}$$

Each coefficient is a nonnegative integer less than 10; hence the digit sum of the product is

$$\begin{aligned} &(a_k + \cdots + a_1 + a_0 - 1) + 9(n - 1) \\ &\qquad + 10 - 2(a_k + \cdots + a_0) + (a_k + \cdots + a_0) = 9n. \end{aligned}$$

Let $m = p_1 p_2 \ldots p_r$ with $p_1, \ldots, p_r$ primes not necessarily distinct. We define

$$c_i = 9N(p_i) - S(p_i) - 9, \text{ for } 1 \leqslant i \leqslant r,$$

let

$$A = \{c_i \mid c_i > 0,\ 1 \leqslant i \leqslant r\},$$

and let n_0 be the number of integers in A.

Lemma 2: $S_p(m) < 9N(m) - \sum_A c_i - .54(r - n_0)$.

The proof involves partitioning the prime factors of m in accordance with their digit sums. Since the result is essentially a refinement of Theorem 1 in [2] (replacing c_i by the number 1 yields that theorem), and the proof is similar, we omit it here.

The above lemma is useful only if some, but not all, of the prime factors of m are known, or, if a lower bound (the higher, the better) on the number of factors of m is known.

3. POWERFUL AND k-SMITH NUMBERS

Theorem 1: There exist infinitely many k-Smith numbers which are not powerful numbers, for each positive integer k.

Proof: Let $n = 2u \not\equiv 0 \pmod{11}$. We have shown in [2] that there exists an integer $b \geqslant 1$ and an integer t belonging to the set $\{2, 3, 4, 5, 7, 8, 15\}$ such that $m = t(10^n - 1) \cdot 10^b$ is a Smith number. Since

$$\begin{aligned} 10^{2u} - 1 &= (10^2 - 1)(10^{2(u-1)} + \cdots + 10^2 + 1) \\ &\equiv 9 \cdot 11 \cdot u \pmod{11}, \end{aligned}$$

it is clear that $11 \mid m$ and $11^2 \nmid m$; hence, m is not a powerful number.

Theorem 2: These exist infinitely many square k-Smith numbers and infinitely many nonsquare powerful k-Smith numbers, for $k > 1$.

Proof: Let $m = (10^n - 1)^2$ and $n = 4u$, u any positive integer. Since $10^4 - 1$ divides $10^{4u} - 1$, $11^2 \cdot 101^2 \mid m$. Setting $p_1 = p_2 = 11$ and $p_3 = p_4 = 101$, we have

$$c_1 = c_2 = 9 \cdot 2 - 2 - 9 = 7 \quad\text{and}\quad c_3 = c_4 = 9 \cdot 3 - 2 - 8 = 16;$$

thus, by Lemma 2, $S_p(m) < 18n - 46$. Let $h = 18n - S_p(m) > 46$. We define

$$T_1 = \{5^3, 2, 2^5, 5^5, 5, 11^3, 2^3 \cdot 5^3\}$$

and

$$T_2 = \{3^4 \cdot 5^2, 15^2, 5^2, 2^2, 2^2 \cdot 3^2 \cdot 17^2, 3^2 \cdot 7^2, 2^4 \cdot 3^2\},$$

and observe that

$$\{S_p(t) \mid t \in T_1\} = \{15, 2, 10, 25, 5, 6, 21\}$$

and

$$\{S_p(t) \mid t \in T_2 = \{22, 16, 10, 4, 26, 20, 14\}$$

are complete residue systems (mod 7).

It follows that there exists an element t in either of T_1 and T_2 such that

$$S_p(t) \equiv h + (k - 2) \cdot 9n \pmod 7,\ k \geqslant 2.$$

Since $h + (k - 2) \cdot 9n > 46$ and $S_p(t) \leqslant 26$, we have

$$S_p(t) = h + (k - 2) \cdot 9n - 7b, \text{ for } b > 2.$$

Let $M = t(10^n - 1)^2 \cdot 10^b$; M is clearly a powerful number. Noting that the hypotheses of Lemma 1 are satisfied, we have $S(M) = 9n$. Thus,

$$\begin{aligned} S_p(M) &= S_p(t) + S_p((10^n - 1)^2) + S_p(10^b) \\ &= [h + (k - 2) \cdot 9n - 7b] + (18n - h) + 7b \\ &= 9kn = kS(M). \end{aligned}$$

This shows that M is a powerful k-Smith number. Now, $m = (10^n - 1)^2$ implies that

$$S_p(m) = 2S_p((10^n - 1))$$

is an even integer. We observe that this implies that h is even, and, since $n = 4u$, that b is even. Since each element of T_1 contains an odd power of a prime, and each element of T_2 is a square, it follows that M is a square if $t \in T_2$, and a nonsquare if $t \in T_1$. Q.E.D.

4. SOME OPEN QUESTIONS

It seems very likely that there exist infinitely many powerful Smith numbers, both squares and nonsquares, i.e., that Theorem 2 is true also when $k = 1$. It would be interesting to know, too, whether there are infinitely many k-Smith numbers which are n^{th} powers of integers for n greater than 2.

Several questions whose answers would provide additional insight into the distribution of k-Smith numbers, but which would appear to be more difficult to answer are also readily suggested: Are there infinitely many consecutive k-Smith numbers for any k (or for every k)? Or, more generally, do infinitely many representations of any integer n exist as the difference of k-Smith numbers for any k? Does every integer have at least one such representation? Although we have not examined an extensive list of Smith numbers, we have found among the composite integers less than 1000, for example, representations of n as the difference of Smith numbers for $n = 2, 3, 4, 5, 6, 7$, and, of course, many larger values of n. We conjecture that every integer is so representable.

Powerful k-Smith numbers occur, of course, much less frequently. Among the integers less than 1000, there are ten: 4, 27, 121, 576, 648, and 729 are powerful Smith numbers, and 32, 361, 200, and 100 are powerful k-Smith numbers for $k = 2, 2, 9$, and 14, respectively. Unexpectedly, however, the frequency with which Smith numbers occur among the powerful numbers less than 1000 is nearly five times as great as the frequency of occurrence among the composite integers less than 1000 which are not powerful. Is this related to the smallness of our sample, or is there another explanation? Finally, in view of the fact that there exist infinitely many representations of every integer as the difference of two powerful numbers [3], we ask: "Which integers are representable as the difference of powerful k-Smith numbers?"

Our thanks to the referee for his helpful suggestions.

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