A NOTE ON DIVISIBILITY SEQUENCES

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In [1], Marshall Hall defined U_n to be a *divisibility sequence* if $U_m | U_n$ whenever $m | n$. Well-known examples of such sequences include geometric sequences and the Fibonacci numbers and their various generalizations (see [2], [3], and the references therein). The purpose of this note is to prove the following theorem.

Theorem: Let U_n be the sequence generated by the recurrence relation

$$U_{n+2} = aU_{n+1} + bU_n,$$

with a, b nonzero integers satisfying $a^2 + 4b = 0$. Then U_n is a nongeometric divisibility sequence if and only if $U_0 = 0$.

Proof: The Binet formula for the sequence U_n is given by

$$U_n = \left(\frac{a}{2}\right)^n (c_1 + c_2 n).$$

If $U_0 = 0$, then $c_1 = U_0 = 0$, $U_n = (a/2)^n c_2 n$, and U_n is a (nongeometric) divisibility sequence.

Conversely, suppose $c_1 = U_0 \neq 0$ and that $U_m | U_n$ whenever $m | n$, i.e., suppose

$$c_1 + c_2 m \left| \left(\frac{a}{2}\right)^{n-m} (c_1 + c_2 n) \right. \text{ whenever } m | n.$$

Replace m by $c_1 a_0 m$, n by $c_1 a_0 n$, and let $a_0 = a/2$ and $e = c_1 a_0 n - c_1 a_0 m$. Then

$$c_1 + c_2 c_1 a_0 m | a_0^e (c_1 + c_1 c_2 a_0 n) \text{ whenever } m | n.$$

Therefore,

$$1 + c_2 a_0 m | a_0^e (1 + c_2 a_0 n) \text{ whenever } m | n.$$

If $e \leqslant 0$, then

$$1 + c_2 a_0 m | 1 + c_2 a_0 n$$

is immediate, while if $e > 0$, since $\gcd(1 + c_2 a_0 m, a_0) = 1$, we also have

$$1 + c_2 a_0 m | 1 + c_2 a_0 n \text{ whenever } m | n.$$

Letting $m = 1$, $n = 2$, gives

$$1 + c_2 a_0 | 1 + 2c_2 a_0 \text{ or } 1 + c_2 a_0 | c_2 a_0.$$

Since $\gcd(1 + c_2 a_0, c_2 a_0) = 1$, it follows that $1 + 2c_2 a_0 = \pm 1$. Hence, either $c_2 a_0 = 0$ or $c_2 a_0 = -2$. If $c_2 a_0 = 0$, then $c_2 = 0$, since $a_0 \neq 0$ by assumption, and we have the geometric sequence $c_1 (a/2)^n$. On the other hand, if $c_2 a_0 = -2$, then we have

$$1 - 2m | 1 - 2n \text{ whenever } m | n,$$

which is false for $m = 2$, $n = 4$.

REFERENCES

1. Marshall Hall. "Divisibility Sequences of 3rd Order." *Amer. J. Math.* 58 (1936):577-84.
2. Clark Kimberling. "Divisibility Properties of Recurrent Sequences." *The Fibonacci Quarterly* 14, no. 4 (1976):369-76.
3. Clark Kimberling. "Generating Functions of Linear Divisibility Sequences." *The Fibonacci Quarterly* 18, no. 3 (1980):193-208.

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