A NOTE ON THE PELL EQUATION

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1. INTRODUCTION

The *Pellian sequence* $\{x_n, n = 1, 2, 3, \ldots\}$ is defined by the rule: x_n is the least positive integer x such that $nx^2 + 1$ is the square of an integer; if no such x exists, x_n is taken to be 0. Briefly, x_n is the least positive solution to the Pell equation $nx^2 + 1 = y^2$. The sequence behaves irregularly; the first few terms are

$$0, 2, 1, 0, 4, 2, 3, 1, 0, 6, 3, 2, 180, 4,$$

while $x_{61} = 1766319049$. It is easy to see that if n is a perfect square, then $x_n = 0$. The converse is also true: it is shown in [2] that for positive nonsquare n, if $\sqrt{n}$ has continued fraction expansion $[a_0, \overline{a_1, \ldots, a_k}]$, then the convergent p_{2k-1}/q_{2k-1} provides a solution $x = q_{2k-1}$, $y = p_{2k-1}$ to the Pell equation $nx^2 + 1 = y^2$ ([2] also serves as a good reference for terminology and facts about continued fractions used in Section 3 of this note). It is also easy to show that $x_n = 1$ if and only if n is one less than a square. In this note, a method will be described which produces all the occurrences of any integer $m > 1$ in the Pellian sequence.

2. POSSIBLE OCCURENCES OF m

It is not difficult to restrict the possible occurences of m in the Pellian sequence to a small list. The method as given in [1] is as follows:

Suppose m is an odd integer greater than 1 and that $x_n = m$. Say $nm^2 + 1 = y^2$ for a positive integer y. Since $nm^2 = (y - 1)(y + 1)$, and m is odd, while $y - 1$ and $y + 1$ share no common odd factors, there must be positive integers a, b with $(a, b) = 1$, $m = ab$, and such that $a^2 \mid (y + 1)$ and $b^2 \mid (y - 1)$. Hence,

$$n = (y^2 - 1)/m^2 = ((y + 1)/a^2)((y - 1)/b^2).$$

If m is even, write $m = 2^e M$ with M odd. In this case, if $nm^2 + 1 = y^2$, then y must be odd and so

$$n2^{2e-2}M^2 = ((y + 1)/2)((y - 1)/2).$$

The factors on the right are consecutive integers. It follows that

$$m/2 = 2^{e-1}M = ab$$

with $(a, b) = 1$ and such that $a^2 \mid (y + 1)/2$ and $b^2 \mid (y - 1)/2$. Thus,

$$n = ((y + 1)/2a^2)((y - 1)/2b^2).$$

So the only possible occurrences of m in the Pellian sequence are found as follows:

1. For odd m write m as a product ab with $(a, b) = 1$ in all possible ways. For even m write $m/2$ as a product ab with $(a, b) = 1$ in all possible ways.

2. For each such factorization ab find the positive solutions to

$y \equiv -1 \pmod{a^2}$

$y \equiv 1 \pmod{b^2}$

if m is odd, or to

$y \equiv -1 \pmod{2a^2}$

$y \equiv 1 \pmod{2b^2}$

if m is even.

Then m can occur in the Pellian sequence only for the numbers $n = (y^2 - 1)/m^2$.

For example, if $m = 35$, there are four systems to solve:

1. $y \equiv -1 \pmod{1^2}$, $y \equiv 1 \pmod{35^2}$
2. $y \equiv -1 \pmod{5^2}$, $y \equiv 1 \pmod{7^2}$
3. $y \equiv -1 \pmod{7^2}$, $y \equiv 1 \pmod{5^2}$
4. $y \equiv -1 \pmod{35^2}$, $y \equiv 1 \pmod{1^2}$

The solutions are, respectively,

1. $y = 1 + 35^2t$,
2. $y = 99 + 35^2t$,
3. $y = 1126 + 35^2t$,
4. $y = 1224 + 35^2t$,

each with $t \geqslant 0$.

Each solution y proivdes a candidate $n = (y^2 - 1)/35^2$, where $x_n = 35$ is possible. These candidates for the four solution sets are, respectively (with $t \geqslant 0$),

1. $(2 + 35^2t)t = 0, 1227, 4904, \ldots,$
2. $(4 + 7^2t)(2 + 5^2t) = 8, 1431, 5304, \ldots,$
3. $(23 + 5^2t)(45 + 7^2t) = 1035, 4512, 10439, \ldots,$
4. $(1 + t)(1224 + 35^2t) = 1224, 4896, 11019, \ldots .$

In fact, x_n is 35 for all the listed values of n except the 0 of solution 1 (x_0 is not even defined) and the 8 of solution 2 ($x_8 = 1$ since 8 is one less than a square). Thus, while the method produces all possible occurrences of m in the Pellian sequence, some exceptional values of n can creep into the lists.

3. EXCEPTIONAL VALUES

When m is odd, the two trivial factorizations of m,

$$m = (1)(m) \quad \text{and} \quad m = (m)(1),$$

give exceptional values of n which are easy to determine. For the first factorization, the system to solve is

$y \equiv -1 \pmod{1^2}$
$y \equiv 1 \pmod{m^2}$,

with solutions $y = 1 + m^2t$, $t \geqslant 0$, which yields candidates

$$n = (y^2 - 1)/m^2 = (2 + m^2t)t.$$

Of course $t = 0$ gives an exceptional value of n. However, all other values of t are good. To see that is so, it must be shown for each $t > 0$ that, if x is a positive integer and $(2 + m^2t)tx^2 + 1 = y^2$, then $x \geqslant m$. From $(2 + m^2t)tx^2 + 1 = y^2$, it follows that

$$2tx^2 + 1 = y^2 - (mtx)^2 \geqslant (mtx + 1)^2 - (mtx)^2 = 2mtx + 1,$$

which shows $x \geqslant m$.

The same reasoning shows that the system

$$y \equiv -1 \pmod{m^2}$$
$$y \equiv 1 \pmod{1^2}$$

yields no exceptional values of n.

Similarly, for even m, the factorization $(1)(m/2)$ of $m/2$ yields one exceptional value of n (namely, $n = 0$), while the factorization $(m/2)(1)$ gives no exceptional values.

For the nontrivial factorizations of m, the exceptional values will be determined by noting a peculiar feature of the continued fraction expansions of $\sqrt{n}$ for the candidate n values produced by each of the systems: the expansions all share common "middle terms." For example, looking at the solutions to system 2 in the example above, the following CFEs are found:

$$\sqrt{8} = [2, \overline{1, 4}] = [2, \overline{1, 4, 1, 4, 1, 4}];$$
$$\sqrt{1431} = [37, \overline{1, 4, 1, 4, 74}];$$
$$\sqrt{5304} = [72, \overline{1, 4, 1, 4, 1, 144}].$$

To see why this is so, let us suppose m is odd and $m = ab$, with $a, b > 1$, $(a, b) = 1$. Let Y be the least positive solution of

$$y \equiv -1 \pmod{a^2}$$
$$y \equiv 1 \pmod{b^2},$$

so that all positive solutions are given by $y = Y + m^2t$, $t \geqslant 0$. For each $t \geqslant 0$, put

$$n_t = ((Y + m^2t)^2 - 1)/m^2,$$

the t^{th} candidate n. If it is observed that

$$[\sqrt{n_t}] = [\sqrt{(Y + m^2t)^2 - 1}/m] = [[\sqrt{(Y + m^2t)^2 - 1}]/m]$$
$$= [(Y + m^2t - 1)/m] = [Y/m] + mt,$$

where $[\cdot]$ denotes the greatest integer function, it is not difficult to verify that the sequence $\sqrt{n_t} - [\sqrt{n_t}]$, $t = 0, 1, \ldots$ is monotone increasing and converges to $Y/m - [Y/m]$. Thus, for all $t \geqslant 1$, we have

$$\sqrt{n_0} - [\sqrt{n_0}] < \sqrt{n_t} - [\sqrt{n_t}] < Y/m - [Y/m].$$

Now, $x = m$, $y = Y$ is certainly a solution to the Pell equation $n_0x^2 + 1 = y^2$, and, consequently, Y/m must be a convergent of the CFE of $\sqrt{n_0}$; in fact, it can be said that

$$\sqrt{n_0} = [q_0, \overline{q_1, \ldots, q_k, 2q_0}],$$

where k is odd, and $q_0 = [Y/m]$, since $[Y/m]$ is the greatest integer in $\sqrt{n_0}$ and, finally, Y/m has CFE $[q_0, q_1, \ldots, q_k]$. The period of the expansion of $\sqrt{n_0}$ is not necessarily $k + 1$, but must be some divisor of $k + 1$. In addition, it is known that $2q_0$ is the largest integer appearing in the CFE of $\sqrt{n_0}$.

So the CFEs of

$$\sqrt{n_0} - [\sqrt{n_0}] = [0, q_1, \ldots, q_k, \ldots]$$

and

$$Y/m - [Y/m] = [0, q_1, \ldots, q_k]$$

are identical out to the entry q_k. Since the numbers $\sqrt{n_t} - [\sqrt{n_t}]$ are trapped between these two values, they also must have continued fraction expansions which begin with $[0, q_1, q_2, \ldots, q_k]$. Furthermore, since $x = m$ certainly provides a solution to the Pell equation $n_t x^2 + 1 = y^2$, it follows that the CFE of $\sqrt{n_t}$ has the form

$$[Q, \overline{q_1, \ldots, q_k, 2Q}], \text{ where } Q = [\sqrt{n_t}].$$

Since the values $q_1, q_2, \ldots, q_k$ are all less than $2q_0$, and so certainly less than $2Q$, it must be that the period of the CFE of $\sqrt{n_t}$ is exactly $k + 1$; hence, m is the least positive x that satisfies the Pell equation $n_t x^2 + 1 = y^2$, which proves that m occurs in the Pellian sequence at every n_t except, possibly, the value n_0.

In a similar fashion, it is found for even m that each nontrivial factorization of m yields at most one exceptional value of n, namely the value

$$n_0 = (Y^2 - 1)/m^2,$$

where Y is the least positive solution for the system.

Thus, the following theorem has been established.

Theorem 1: For $m > 1$ odd, write $m = ab$ with $(a, b) = 1$, and let Y be the least positive solution of the system

$$\begin{aligned} y &\equiv -1 \pmod{a^2} \\ y &\equiv 1 \pmod{b^2}. \end{aligned} \tag{1}$$

Then $m = x_n$, the n^{th} term of the Pellian sequence, where n is given by

$$n = ((Y + m^2 t)^2 - 1)/m^2, \text{ for all } t \geq 1,$$

and possibly for $t = 0$ as well. This accounts for all occurrences of m.

For $m > 1$ even, write $m/2 = ab$ with $(a, b) = 1$, and let Y be the least positive solution of the system

$$\begin{aligned} y &\equiv -1 \pmod{2a^2} \\ y &\equiv 1 \pmod{2b^2}. \end{aligned} \tag{2}$$

Then $m = x_n$, the n^{th} term of the Pellian sequence, where n is given by

$$n = ((Y + m^2 t)^2 - 1)/m^2, \text{ for all } t \geq 1,$$

and possibly for $t = 0$ as well. This accounts for all occurrences of m.

It is natural to ask exactly when $t = 0$ will yield an exceptional n. While a general solution of this problem appears to be difficult, for some particular nontrivial facotrizations ab of m (or $m/2$), the answer can be provided. For example, when m is odd, a factorization of the form $a(a + 2)$ always gives an exceptional value of n (as was seen for the case $35 = 5 \cdot 7$ in the earlier example). To see why this is true, suppose $a = 2k + 1$ and $b = 2k + 3$. The least positive solution to the system

$$\begin{aligned} y &\equiv -1 \pmod{a^2} \\ y &\equiv 1 \pmod{b^2} \end{aligned}$$

is

$$Y = (k + 2)(2k + 1)^2 - 1 = k(2k + 3)^2 + 1,$$

which provides us with

$$n = k(k + 2) = (k + 1)^2 - 1,$$

always one less than a square. Hence, $x_n = 1$, and this n is exceptional. However, such factorizations do not account for all exceptional values of n. For

$m = 1197 = 19 \cdot 63$, the least positive solution to

$$y \equiv -1 \pmod{19^2}$$
$$y \equiv 1 \pmod{63^2}$$

is $Y = 3970$, which yields $n = 11$. But $x_{11} = 3$ and not 1197. Likewise, it can be shown that if m is even and $m/2$ is factored as $(m/4)(2)$ (assuming m is a multiple of 4), then for the n produced, $x_n = 2$, and not m. Again there are other factorizations which yield exceptional values of n.

REFERENCES

1. S. P. Kaler. Properties of the Pellian Sequence." Masters Thesis. University of North Dakota, 1985.
2. W. J. LeVeque. *Fundamentals of Number Theory*. Reading, Mass.: Addison-Wesley, 1977.

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