A NOTE ON A GENERALIZATION OF EULER'S ϕ FUNCTION

LASZLO TOTH
3900 Satu Mare, str. N. Golescu 5, Romania

(Submitted August 1985)

P. G. Garcia and Steve Ligh [3] introduced the following generalization of the Euler function $\phi(n)$: For an arithmetic progression

$$D(s, d, n) = \{s, s + d, \ldots, s + (n - 1)d\},$$

where $(s, d) = 1$, let $\phi(s, d, n)$ denote the number of elements in $D(s, d, n)$ that are relatively prime to n. Observe that $\phi(1, 1, n) \equiv \phi(n)$.

Garcia and Ligh showed that $\phi(s, d, n)$ is multiplicative in n, i.e., for $(m, n) = 1$, we have

$$\phi(s, d, mn) = \phi(s, d, m)\phi(s, d, n)$$

(cf. [3], Theorem 1), and deduced the formula:

$$\phi(s, d, p^k) = \begin{cases} p^k\left(1 - \frac{1}{p}\right), & \text{if } p \nmid d, \\ p^k, & \text{if } p \mid d, \end{cases} \tag{1}$$

(cf. [3], Lemma 2).

The aim of this note is to establish an asymptotic formula for the summatory function of $\phi(s, d, n)$ using an elementary method.

Let μ denote the Möbius function, I the Dirac function, for which

$$I(n) = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$$

and let I_d be the arithmetic function defined by $I_d(n) = I((n, d))$. We need the following result, which is the generalization of the familiar Dedekind-Liouville evaluation of $\phi(n)$:

$$\phi(n) = \sum_{er = n} \mu(e)r.$$

Lemma 1: $\phi(s, d, n) = \sum_{er = n} \mu(e)I_d(e)r \equiv \sum_{\substack{er = n \\ (e, d) = 1}} \mu(e)r. \quad (2)$

Proof: The functions μ, I_d, and $\mu \cdot I_d$ are multiplicative [moreover, I_d is totally multiplicative, i.e., $I_d(mn) = I_d(m)I_d(n)$ for arbitrary m and n] and so the right-hand sum, being the Dirichlet convolution of two multiplicative functions, is also multiplicative. It has been noted that $\phi(s, d, n)$ is multiplicative; thus, it is enough to verify the above identity for $n = p^k$. We have:

$$\sum_{er = p^k} \mu(e)I_d(e)r = \begin{cases} p^k - p^{k-1} = p^k\left(1 - \frac{1}{p}\right), & \text{if } p \nmid d \\ p^k, & \text{if } p \mid d \end{cases}$$

$$= \phi(s, d, p^k) \quad \text{by (1).}$$

Corollary 1: $\sum_{er=n} I_d(e)\phi(s, d, r) \equiv \sum_{\substack{er=n\\(e, d)=1}} \phi(s, d, r) = n.$ (3)

Proof: By Lemma 1 we have $\phi(s, d, n) = \mu \cdot I_d * E$, where $E(n) = n$ and $*$ denotes the Dirichlet convolution. Thus,

$$I_d * \phi(s, d, n) = I_d * \mu \cdot I_d * E,$$

and, using the distributivity property of the totally multiplicative functions (see, for example, [4], Theorem 1):

$$I_d * \phi(s, d, n) = I_d(U * \mu) * E,$$

where $U(n) = 1$ and $(U * \mu) * E = I * E = E$. Hence,

$$I_d * \phi(s, d, n) = E$$

and the proof is complete.

Remark 1: The author thanks the referee for the following direct proof of (3):

We write n as $n = PQ$, $(P, Q) = 1$, where $(P, d) = 1$ and $(Q, d) > 1$ or $Q = 1$. By the multiplicative property of $\phi(s, d, n)$,

$$\phi(s, d, n) = \phi(s, d, P)\phi(s, d, Q) = \phi(P)Q$$

(cf. [5], Lemma 2). Thus,

$$\sum_{\substack{er=n\\(e, d)=1}} \phi(s, d, r) = \sum_{J|P} \phi(s, d, jQ) = \sum_{J|P} \phi(s, d, j)\phi(s, d, Q)$$
$$= Q\sum_{J|P} \phi(j) = PQ = n.$$

Remark 2: Ligh and Garcia have obtained a formula for $\sum_{r|n} \phi(s, d, r)$ (see [5], Theorem 2).

Let $J(n)$ denote the Jordan totient function of second order,

$$J(n) = n^2 \prod_{p|n}\left(1 - \frac{1}{p^2}\right) \quad \text{(see [2], p. 147).}$$

Lemma 2 (cf. [1], Lemma 5.1):

$$\sum_{n=1}^{\infty} \frac{\mu(n)I_d(n)}{n^2} = \frac{6d^2}{\pi^2 J(d)} \tag{4}$$

Proof: The series is absolutely convergent and the general term is a multiplicative function of n; thus, it can be expanded into an infinite product of the Euler type (see [2], § 17.4):

$$\sum_{n=1}^{\infty} \frac{\mu(n)I_d(n)}{n^2} = \prod_p\left(\sum_{i=0}^{\infty} \frac{\mu(p^i)I_d(p^i)}{p^{2i}}\right) = \prod_{p\nmid d}\left(1 - \frac{1}{p^2}\right)$$
$$= \frac{\prod_p\left(1 - \frac{1}{p^2}\right)}{\prod_{p|d}\left(1 - \frac{1}{p^2}\right)} = \frac{d^2}{\zeta(s)J(d)} = \frac{6d^2}{\pi^2 J(d)},$$

where $\zeta(s)$ is the Riemann Zeta function.

We shall use the following well-known estimates.

Lemma 3: $$\sum_{n \leqslant x} n = \frac{x^2}{2} + O(x) \tag{5}$$

$$\sum_{n \leqslant x} \frac{1}{n} = O(\log x) \tag{6}$$

$$\sum_{n > x} \frac{1}{n^2} = O\left(\frac{1}{x}\right) \tag{7}$$

Theorem: $$\sum_{n \leqslant x} \phi(s, d, n) = \frac{3d^2}{\pi^2 J(d)} x^2 + O(x \log x). \tag{8}$$

Proof: Using (2) and (5), we have:

$$\begin{aligned}
\sum_{n \leqslant x} \phi(s, d, n) &= \sum_{er \leqslant x} \mu(e) I_d(e) r = \sum_{e \leqslant x} \mu(e) I_d(e) \sum_{r \leqslant x/e} r \\
&= \sum_{e \leqslant x} \mu(e) I_d(e) \left\{ \frac{x^2}{2e^2} + O\left(\frac{x}{e}\right) \right\} = \frac{x^2}{2} \sum_{e \leqslant x} \frac{\mu(e) I_d(e)}{e^2} + O\left(x \sum_{e \leqslant x} \frac{1}{e}\right) \\
&= \frac{x^2}{2} \sum_{e=1}^{\infty} \frac{\mu(e) I_d(e)}{e^2} + O\left(x^2 \sum_{e > x} \frac{1}{e^2}\right) + O\left(x \sum_{e \leqslant x} \frac{1}{e}\right).
\end{aligned}$$

And now, by (4), (7), and (6),

$$\begin{aligned}
\sum_{n \leqslant x} \phi(s, d, n) &= \frac{x^2}{2} \cdot \frac{6d^2}{\pi^2 J(d)} + O(x) + O(x \log x) \\
&= \frac{3d^2}{\pi^2 J(d)} x^2 + O(x \log x).
\end{aligned}$$

Corollary 2: The average order of $\phi(s, d, n)$ is $\dfrac{6d^2}{\pi^2 J(d)} n$.

Proof: From (8), we have

$$\frac{1}{x} \sum_{n \leqslant x} \phi(s, d, n) \sim \frac{1}{x} \sum_{n \leqslant x} f_d(n), \text{ where } f_d(n) = \frac{6d^2}{\pi^2 J(d)} n.$$

For $d = 1$, we reobtain Mertens' formula:

Corollary 3: $$\sum_{n \leqslant x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

REFERENCES

1. E. Cohen. "Arithmetical Functions Associated with the Unitary Divisors of an Integer." *Math z.* *74* (1960):66-80.
2. L. E. Dickson. *History of the Theory of numbers.* Vol. I. New York: Chelsea, 1952.
3. P. G. Garcia & Steve Ligh. "A Generalization of Euler's ϕ Function." *The Fibonacci Quarterly 21*, no. 1 (1983):26-28.
4. J. Lambek. "Arithmetical Functions and Distributivity." *Amer. Math. Monthly 73* (1966):969-73.

5. Steve Ligh & P. G. Garcia. "A Generalization of Euler's ϕ Function, II." *Jath. Japonica* 30 (1985):519-22.

◆◇◆◇◆

Announcement

THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 25-29, 1988
Department of Mathematics, University of Pisa
Pisa, Italy

International Committee

Horadam, A.F. (Australia), *Co-Chairman*
Philippou, A.N. (Greece), *Co-Chairman*
Ando, S. (Japan)
Bergum, G.E. (U.S.A.)
Johnson, M.B. (U.S.A.)
Kiss, P. (Hungary)
Schinzel, Andrzej (Poland)
Tijdeman, Robert (The Netherlands)
Tognetti, K. (Australia)

Local Committee

Robert Dvornicich, *Chairman*
Piero Filipponi
Alberto Perelli
Carlo Viola
Umberto Zannier

FIBONACCI'S STATUE

Have you ever seen Fibonacci's portrait? This photo is a close-up of the head of the statue of Leonardo Pisano in Pisa, Italy, taken by Frank Johnson in 1978.

Since Fibonacci's statue was difficult to find, here are the directions from the train station in Pisa (about 8 blocks to walk): Cross Piazza Vitt. Em. II, bearing right along Via V. Croce to Piazza Toniolo, and then walk through the Fortessa. The statue is found within Fortezza Campo Santo off Lungarno Fibonacci or Via Fibonacci along the Arno River at Giardino Scotto (Teatro Estivo).

CALL FOR PAPERS

The THIRD INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at The University of Pisa, Pisa, Italy, from July 25-29, 1988. This conference is sponsored jointly by The Fibonacci Association and The University of Pisa.

Papers on all branches of mathematics and science related to the Fibonacci numbers and their generalizations are welcome. Abstracts are to be submitted by March 15, 1988. Manuscripts are requested by May 1, 1988. Abstracts and manuscripts should be sent to G.E. Bergum (address below). Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by June 15, 1988. All talks should be limited to one hour or less.

For further information concerning the conference, please contact Gerald Bergum, The Fibonacci Quarterly, Department of Mathematics, South Dakota State University, P.O. Box 2220, Brookings, South Dakota 57007-1297.