ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-604 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*

Let c be a fixed number and $u_{n+2} = cu_{n+1} + u_n$ for n in $N = \{0, 1, 2, \ldots\}$. Show that there exists a number h such that

$$u_{n+4}^2 = hu_{n+3}^2 - hu_{n+1}^2 + u_n \text{ for } n \text{ in } N.$$

B-605 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{i=1}^{n} L_{2n+2i-1}.$$

Determine the positive integers n, if any, for which $S(n)$ is prime.

B-606 *Proposed by L. Kuipers, Sierre, Switzerland*

Simplify the expression

$$L_{n+1}^2 + 2L_{n-1}L_{n+1} - 25F_n^2 + L_{n-1}^2.$$

B-607 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let

$$C_n = \sum_{k=0}^{n} \binom{n}{k} F_k L_{n-k}.$$

Show that $C_n/2^n$ is an integer for n in $\{0, 1, 2, \ldots\}$.

B-608 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

For $k = \{2, 3, \ldots\}$ and n in $N = \{0, 1, 2, \ldots\}$, let

$$S_{n,k} = \frac{1}{k}\sum_{j=n}^{n+k-1} F_j^2$$

denote the quadratic mean taken over k consecutive Fibonacci numbers of which the first is F_n. Find the smallest such $k \geqslant 2$ for which $S_{n,k}$ is an integer for all n in N.

B-609 *Proposed by Adina DiPorto & Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Find a closed form expression for

$$S = \sum_{k=1}^{n} (kF_k)^2$$

and show that $S_n \equiv n(-1)^n \pmod{F_n}$.

SOLUTIONS

Nondivisors of the L_n

B-580 *Proposed by Valentina Bakinova, Rondout Valley, NY*

What are the three smallest positive integers d such that no Lucas number L_n is an integral multiple of d?

Solution by J. Suck, Essen, Germany

They are 5, 8, 10. Since $1|L_n$, $2|L_0$, $3|L_2$, $4|L_3$, $6|L_6$, $7|L_4$, $9|L_6$, it remains to show that $5\nmid L_n$ and $8\nmid L_n$ for all $n = 0, 1, 2, \ldots$. This follows from the fact that the Lucas sequence modulo 5 or 8 is periodic with period 2, 1, 3, 4 or 2, 1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, respectively.

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, Herta T. Freitag, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, and the proposer.

Third Degree Representations for F

B-581 *Proposed by Antal Bege, University of Cluj, Romania*

Prove that, for every positive integer n, there are at least $[n/2]$ ordered 6-tuples (a, b, c, x, y, z) such that

$$F_n = ax^2 + by^2 - cz^2$$

and each of a, b, c, x, y, z is a Fibonacci number. Here $[t]$ is the greatest integer in t.

Solution by Paul S. Bruckman, Fair Oaks, CA

We first prove the following relations:

$$F_{2n} = F_{2s+1}F_{n-s+1}^2 + F_{2s}F_{n-s}^2 - F_{2s-1}F_{n-s-1}^2; \tag{1}$$

$$F_{2n+1} = F_{2s+2}F_{n-s+1}^2 + F_{2s+1}F_{n-s}^2 - F_{2s}F_{n-s-1}^2, \tag{2}$$

valid for all integers s and n.

Proof of (1) and (2): We use the following relations repeatedly:

$$F_u F_v^2 = \frac{1}{5}(F_{2v+u} - (-1)^u F_{2v-u} - 2(-1)^v F_u), \tag{3}$$

which is readily proven from the Binet formulas and is given without proof.

Multiplying the right member of (1) by 5, we apply (3) to transform the result as follows:

$$\begin{aligned}
&(F_{2n+3} + F_{2n-4s+1} + 2(-1)^{n-s}F_{2s+1}) + (F_{2n} - F_{2n-4s} - 2(-1)^{n-s}F_{2s})\\
&\quad - (F_{2n-3} + F_{2n-4s-1} + 2(-1)^{n-s}F_{2s-1})\\
&= (F_{2n+3} - F_{2n-3} + F_{2n}) + (F_{2n-4s+1} - F_{2n-4s} - F_{2n-4s-1})\\
&\quad + 2(-1)^{n-s}(F_{2s+1} - F_{2s} - F_{2s-1})\\
&= (L_3F_{2n} + F_{2n}) + 0 + 0 = 5F_{2n}.
\end{aligned}$$

This proves (1).

Likewise, multiplying the right member of (2) by 5 yields:

$$\begin{aligned}
&(F_{2n+4} - F_{2n-4s} + 2(-1)^{n-s}F_{2s+2}) + (F_{2n+1} + F_{2n-4s-1} - 2(-1)^{n-s}F_{2s+1})\\
&\quad - (F_{2n-2} - F_{2n-4s-2} + 2(-1)^{n-s}F_{2s})\\
&= (F_{2n+4} - F_{2n-2} + F_{2n+1}) - (F_{2n-4s} - F_{2n-4s-1} - F_{2n-4s-2})\\
&\quad + 2(-1)^{n-s}(F_{2s+2} - F_{2s+1} - F_{2s})\\
&= (L_3F_{2n+1} + F_{2n+1}) - 0 + 0 = 5F_{2n+1}.
\end{aligned}$$

This proves (2).

We may combine (1) and (2) into the single formula:

$$F_n = F_{2s+1+o_n}F_{m-s+1}^2 + F_{2s+o_n}F_{m-s}^2 - F_{2s-1+o_n}F_{m-s-1}^2, \tag{4}$$

where

$$m \equiv [n/2], \quad o_n \equiv (1 - (-1)^n)/2 = \begin{cases} 1, & n \text{ odd},\\ 0, & n \text{ even}.\end{cases}$$

We see that the 6-tuples

$$\begin{aligned}
&(a, b, c, x, y, z)\\
&= (F_{2s+1+o_n}, F_{2s+o_n}, F_{2s-1+o_n}, F_{m-s+1}, F_{m-s}, F_{m-s-1})
\end{aligned} \tag{5}$$

are solutions of the problem, as s is allowed to vary. For at least the values $s = 0, 1, \ldots, m - 1$, different 6-tuples are produced in (5). Hence, there are at least $m = [n/2]$ distinct 6-tuples solving the problem.

Also solved by the proposer.

Zeckendorf Representations

B-582 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

It is known that every positive integer N can be represented uniquely as a sum of distinct nonconsecutive positive Fibonacci numbers. Let $f(N)$ be the number of Fibonacci addends in this representation, $\alpha = (1 + \sqrt{5})/2$, and $[x]$ be the greatest integer in x. Prove that

$$f([\alpha F_n^2]) = [(n + 1)/2] \text{ for } n = 1, 2, \ldots .$$

Solution by L. A. G. Dresel, University of Reading, England

Since

$$F_r^2 - F_{r-2}^2 = (F_r - F_{r-2})(F_r + F_{r-2}) = F_{r-1}L_{r-1} = F_{2(r-1)},$$

we have, summing for *even* values $r = 2t$, $t = 1, 2, \ldots, m$,

$$F_{2m}^2 - 0 = F_{4m-2} + F_{4m-6} + \cdots + F_2,$$

and summing for *odd* values $r = 2t + 1$, $t = 1, 2, \ldots, m$,

$$F_{2m+1}^2 - 1 = F_{4m} + F_{4m-4} + \cdots + F_4.$$

Let $a = \frac{1}{2}(1 + \sqrt{5})$ and $b = \frac{1}{2}(1 - \sqrt{5})$, then

$$aF_{2s} = (a^{2s+1} - ab^{2s})/\sqrt{5} = F_{2s+1} + (b - a)b^{2s}/\sqrt{5} = F_{2s+1} - b^{2s}.$$

Applying the formula for F_{2m}^2, we obtain

$$aF_{2m}^2 = F_{4m-1} + F_{4m-5} + \cdots + F_3 - (b^{4m-2} + b^{4m-6} + \cdots + b^2)$$

and since $0 < (b^2 + b^6 + \cdots + b^{4m-2}) < b^2/(1 - b^4) < 1$, we have

$$[aF_{2m}^2] = F_{4m-1} + F_{4m-5} + \cdots + F_3 - 1.$$

Putting $F_3 - 1 = F_2$, we have a sum of m nonconsecutive Fibonacci numbers. Similarly,

$$aF_{2m+1}^2 = F_{4m+1} + F_{4m-3} + \cdots + F_5 + a - (b^{4m} + \cdots + b^8 + b^4),$$
$$0 < (b^4 + b^8 + \cdots + b^{4m}) < b^4/(1 - b^4) < b^2,$$

and $\quad 1 < a - b^2 < 2,$

so that

$$[aF_{2m+1}^2] = F_{4m+1} + F_{4m-3} + \cdots + F_5 + F_1,$$

which is the sum of $(m+1)$ nonconsecutive Fibonacci numbers. Finally, for $n = 1$, we have

$$[aF_1^2] = 1 = F_1.$$

Thus, in all cases, we have

$$f([aF_n^2]) = [(n + 1)/2], \; n = 1, 2, \ldots .$$

Also solved by Paul S. Bruckman, L. Kuipers, J. Suck, and the proposer.

Recursion for a Triangle of Sums

B-583 *Proposed by Dorin Andrica, University of Cluj-Napoca, Romania*

For positive integers n and s, let

$$S_{n,s} = \sum_{k=1}^{n} \binom{n}{k} k^s.$$

Prove that $S_{n,s+1} = n(S_{n,s} - S_{n-1,s})$.

Solution by J.-J. Seiffert, Berlin, Germany

With $\binom{n-1}{n} := 0$ and $\binom{n}{k} - \binom{n-1}{k} = \frac{k}{n}\binom{n}{k}$, we obtain

$$S_{n,s} - S_{n-1,s} = \sum_{k=1}^{n}\left(\binom{n}{k} - \binom{n-1}{k}\right)k^s = \frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}k^{s+1} = \frac{1}{n}S_{n,s+1}.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Russell Euler, Piero Filipponi & Odoardo Brugia, Herta T. Freitag, Fuchin He, Joseph J. Kostal, L. Kuipers, Carl Libis, Bob Prielipp, J. Suck, Nicola Treitzenberg, Paul Tzermias, Tad P. White, and the proposer.

Product of Exponential Generating Functions

B-584 *Proposed by Dorin Andrica, University of Cluj-Napoca, Romania*

Using the notation of B-583, prove that

$$S_{m+n,s} = \sum_{k=0}^{s} \binom{s}{k} S_{m,k} S_{n,s-k}.$$

Solution by Heinz-Jürgen Seiffert, Berlin, Germany

The stated equation is not meaningful if one uses the notation of B-583. (To see this, put $s = 0$.) But such an equation can be proved for

$$S_{n,s} := \sum_{k=0}^{n}\binom{n}{k}k^s, \tag{1}$$

with the usual convention $0^0 := 1$. Consider the function

$$F(x, n) := \sum_{s=0}^{\infty} S_{n,s}\frac{x^s}{s!}. \tag{2}$$

Since $0 \leqslant S_{n,s} \leqslant 2^n n^s$, the above series converges for all real x. Using (1), one obtains

$$F(x, n) = \sum_{s=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{(kx)^s}{s!} = \sum_{k=0}^{n}\binom{n}{k}\sum_{s=0}^{\infty}\frac{(kx)^s}{s!} = \sum_{k=0}^{n}\binom{n}{k}e^{kx}$$

or

$$F(x, n) = (e^x + 1)^n, \tag{3}$$

which yields

$$F(x, m + n) = F(x, m)F(x, n). \tag{4}$$

Cauchy's product leads to

$$F(x,\ m)F(x,\ n) = \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{S_{m,k}}{k!} \frac{S_{n,s-k}}{(s-k)!} x^s \tag{5}$$

From (2), (4), and (5), and by comparing coefficients, one obtains the equation as stated in the proposal for the $S_{n,s}$ defined in (1).

Also solved by Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, L. A. G. Dresel, L. Kuipers, Fuchin He, J. Suck, Nicola Treitzenberg, Paul Tzermias, Tad P. White, and the proposer.

Combinatorial Interpretation of the F_n

B-585 *Proposed by Constantin Gonciulea & Nicolae Gonciulea, Trian College, Drobeta Turnu-Severin, Romania*

For each subset A of $X = \{1, 2, \ldots, n\}$, let $r(A)$ be the number of j such that $\{j, j+1\} \subseteq A$. Show that

$$\sum_{A \subseteq X} 2^{r(A)} = F_{2n+1}.$$

Solution by J. Suck, Essen, Germany

Let us supplement the proposal by

$$\text{"and } \sum_{n \in A \subseteq X} 2^{r(A)} = F_{2n}\text{."}$$

We now have a beautiful combinatorial interpretation of the Fibonacci sequence. The two identities help each other in the following induction proof.

For $n = 1$, $A = \emptyset$ or X, $r(A) = 0$. Thus, both identities hold here. Suppose they hold for $k = 1, \ldots, n$. Consider $Y := \{1, \ldots, n, n+1\}$. If $\{n, n+1\} \subseteq B \subseteq Y$, $r(B) = r(B \backslash \{n+1\}) + 1$. If $n \notin B \subseteq Y$, $r(B) = r(B \backslash \{n+1\})$. Thus,

$$\sum_{n+1 \in B \subseteq Y} 2^{r(B)} = \sum_{n \in A \subseteq X} 2^{r(A)+1} + \sum_{A \subseteq X \backslash \{n\}} 2^{r(A)} \quad \text{(the last sum is 1 for the step } 1 \to 1+1\text{)}$$

$$= 2F_{2n} + F_{2(n-1)+1} = F_{2n} + F_{2n+1} = F_{2(n+1)},$$

and

$$\sum_{B \subseteq Y} 2^{r(B)} = \sum_{n+1 \in B \subseteq Y} 2^{r(B)} + \sum_{A \subseteq X} 2^{r(A)} = F_{2(n+1)} + F_{2n+1} = F_{2(n+1)\ 1}.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, N. J. Kuenzi & Bob Prielipp, Paul Tzermias, Tad P. White, and the proposer.

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