ON SOME PROPERTIES OF HUMBERT'S POLYNOMIALS

G. V. MILOVANOVIĆ
University of Niš, P.O. Box 73, 18000 Niš, Yugoslavia
G. DJORDJEVIĆ
University of Niš, 16000 Leskovac, Yugoslavia

(Submitted February 1986)

1. INTRODUCTION

In 1921, Humbert [8] defined a class of polynomials $\{\Pi_{n,m}^{\lambda}\}_{n=0}^{\infty}$ by the generating function

$$(1 - mxt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} \Pi_{n,m}^{\lambda}(x)t^n. \tag{1}$$

These satisfy the recurrence relation

$$(n + 1)\Pi_{n+1,m}^{\lambda}(x) - mx(n + \lambda)\Pi_{n,m}^{\lambda}(x) - (n + m\lambda - m + 1)\Pi_{n-m+1,m}^{\lambda}(x) = 0.$$

Particular cases of these polynomials are Gegenbauer polynomials [1]

$$C_n^{\lambda}(x) = \Pi_{n,2}^{\lambda}(x)$$

and Pincherle polynomials (see [8])

$$\mathcal{P}_n(x) = \Pi_{n,3}^{-1/2}(x).$$

Later, Gould [2] studied a class of generalized Humbert polynomials

$$P_n(m, x, y, p, C)$$

defined by

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n, \tag{2}$$

where $m \geqslant 1$ is an integer and the other parameters are unrestricted in general. The recurrence relation for the generalized Humbert polynomials is

$$CnP_n - m(n - 1 - p)xP_{n-1} + (n - m - mp)yP_{n-m} = 0, \quad n \geqslant m \geqslant 1, \tag{3}$$

where we put $P_n = P_n(m, x, y, p, C)$.

In [6], Horadam and Pethe investigated the polynomials associated with the Gegenbauer polynomials

$$C^{\lambda}(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(\lambda)_{n-k}}{k!(n - 2k)!}(2x)^{n-2k}, \tag{4}$$

where $(\lambda)_0 = 1$, $(\lambda)_n = \lambda(\lambda + 1) \ldots (\lambda + n - 1)$, $n = 1, 2, \ldots$. Listing the polynomials of (4) horizontally and taking sums along the rising diagonals, Horadam and Pethe obtained the polynomials denoted by $p_n^{\lambda}(x)$. For these polynomials, they proved that the generating function $G^{\lambda}(x, t)$ is given by

$$G^{\lambda}(x, t) = \sum_{n=1}^{\infty} p_n^{\lambda}(x)t^{n-1} = (1 - 2xt + t^3)^{-\lambda}. \tag{5}$$

Some special cases of these polynomials were considered in several papers (see [3], [4], and [7], for example).

Comparing (5) to (1), we see that their polynomials are Humbert polynomials for $m = 3$, with x replaced by $2x/3$, i.e., $p_{n+1}^{\lambda}(x) = \Pi_{n,3}^{\lambda}(2x/3)$.

2. THE POLYNOMIALS $p_{n,m}^{\lambda}(x)$

In this paper, we consider the polynomials $\{p_{n,m}^{\lambda}\}_{n=0}^{\infty}$ defined by

$$p_{n,m}^{\lambda}(x) = \Pi_{n,m}^{\lambda}(2x/m).$$

Their generating function is given by

$$G_m^{\lambda}(x, t) = (1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,m}^{\lambda}(x)t^n. \tag{6}$$

Note that

$$p_{n,2}^{\lambda}(x) = C_n^{\lambda}(x) \qquad \text{(Gegenbauer polynomials)}$$

and

$$p_{n,3}^{\lambda}(x) = p_{n+1}^{\lambda}(x) \qquad \text{(Horadam-Pethe polynomials).}$$

For $m = 1$, we have

$$G_1^{\lambda}(x, t) = (1 - (2x - 1)t)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,1}^{\lambda}(x)t^n$$

and

$$p_{n,1}^{\lambda}(x) = (-1)^n \binom{-\lambda}{n}(2x - 1)^n = \frac{(\lambda)_n}{n!}(2x - 1)^n.$$

These polynomials can be obtained from descending diagonals in the Pascal-type array for Gegenbauer polynomials (see Horadam [5]).

Expanding the left-hand side of (6), we obtain the explicit formula

$$p_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(\lambda)_{n-(m-1)k}}{k!(n - mk)!}(2x)^{n-mk}. \tag{7}$$

These polynomials can be obtained from (2) by putting $C = y = 1$, $p = -\lambda$, and $x := 2x/m$. Then we have

$$p_{n,m}^{\lambda}(x) = P_n(m, 2x/m, 1, -\lambda, 1).$$

Also, if we put $C = y = m/2$ and $p = -\lambda$, we obtain

$$p_{n,m}^{\lambda}(x) = \left(\frac{2}{m}\right)^{\lambda} P_n(m, x, m/2, -\lambda, m/2).$$

Then, from (3), we get the following recurrence relation

$$np_{n,m}^{\lambda}(x) = (\lambda + n - 1)2xp_{n-1,m}(x) - (n + m(\lambda - 1))p_{n-m,m}(x), \tag{8}$$

for $n \geqslant m \geqslant 1$.

The starting polynomials are

$$p_{n,m}^{\lambda}(x) = \frac{(\lambda)_n}{n!}(2x)^n, \; n = 0, 1, \ldots, m - 1.$$

Remark: For corresponding monic polynomials $\hat{p}^{\lambda}_{n,m}$, we have

$$\hat{p}^{\lambda}_{n,m}(x) = x\hat{p}^{\lambda}_{n-1,m}(x) - b_n\hat{p}^{\lambda}_{n-m,m}(x), \; n \geqslant m \geqslant 1,$$

$$\hat{p}^{\lambda}_{n,m}(x) = x^n, \; 0 \leqslant n \leqslant m-1,$$

where

$$b_n = \frac{(n-1)!}{(m-1)!} \cdot \frac{n + m(\lambda - 1)}{2^m(\lambda + n - m)_m}.$$

The classes of polynomials $\mathbb{P}_{m,\lambda} = \{p^{\lambda}_{n,m}\}^{\infty}_{n=0}$, $m = 2, 3, \ldots$, can be found by repeating the "diagonal functions process," starting from $p^{\lambda}_{n,1}(x)$. Listing the terms of polynomials horizontally,

$$p^{\lambda}_{n,m}(x) = \sum_{k=0}^{[n/m]} a^{\lambda}_{n,m}(k)(2x)^{n-mk}, \quad a^{\lambda}_{n,m}(k) = \frac{(-1)^k(\lambda)_{n-(m-1)k}}{k!(n-mk)!},$$

and taking sums along the rising diagonals, we obtain $p^{\lambda}_{n,m+1}(x)$, because

$$a^{\lambda}_{n-k,m}(k) = (-1)^k \frac{(\lambda)_{n-k-(m-1)k}}{k!(n-k-mk)!} = a^{\lambda}_{n,m+1}(k).$$

3. SOME DIFFERENTIAL RELATIONS

In this section we shall give some differential equalities for the polynomials $p^{\lambda}_{n,m}$. Here, D is the differentiation operator and $p^{\lambda}_{k,m}(x) \equiv 0$ when $k \leqslant 0$.

Theorem 1: The following equalities hold:

$$D^k p^{\lambda}_{n+k,m}(x) = 2^k(\lambda)_k p^{\lambda+k}_{n,m}(x), \tag{9}$$

$$2np^{\lambda}_{n,m}(x) = 2xDp^{\lambda}_{n,m}(x) - mDp^{\lambda}_{n-m+1,m}(x), \tag{10}$$

$$mDp^{\lambda}_{n+1,m}(x) = 2(n + m\lambda)p^{\lambda}_{n,m}(x) + 2x(m-1)Dp^{\lambda}_{n,m}(x), \tag{11}$$

$$2\lambda p^{\lambda}_{n,m}(x) = Dp^{\lambda}_{n+1,m}(x) - 2xDp^{\lambda}_{n,m}(x) + Dp^{\lambda}_{n-m+1,m}(x). \tag{12}$$

Proof: Using the differentiation formula (cf. [2, Eq. (3.5)])

$$D^k_x P_{n+k}(m, x, y, p, C) = (-m)^k k! \binom{p}{k} P_n(m, x, y, p-k, C)$$

we obtain (9).

To prove (10), we differentiate the generating function (6) w.r.t. x and t. Then, elimination $(1 - 2xt + t^m)^{-\lambda-1}$ from the expressions, we find

$$\sum_{n=1}^{\infty} 2np^{\lambda}_{n,m}(x)t^n = (2x - mt^{m-1})\sum_{n=0}^{\infty} Dp^{\lambda}_{n,m}(x)t^n.$$

Equating coefficients of t^n in this identity, we get (10).

By differentiating the recurrence relation (8), with $n + 1$ substituted for n, and using (10), we obtain (11).

Finally, by differentiating the generating function (6) w.r.t. x, replacing $G^{\lambda}_m(x, t)$ by its series expansion in powers of t, and equating coefficients of t^{n+1}, we obtain the relation (12).

4. THE DIFFERENTIAL EQUATION

Let the sequence $(f_r)_{r=0}^n$ be given by $f_r = f(r)$, where

$$f(t) = (n - t)\left(\frac{n - t + m(\lambda + t)}{m}\right)_{m-1}.$$

Also, we introduce two standard difference operators, the forward difference operator Δ and the displacement (or shift) operator E, by

$$\Delta f_r = f_{r+1} - f_r \quad \text{and} \quad Ef_r = f_{r+1},$$

and their powers by

$$\Delta^0 f_r = f_r, \quad \Delta^k f_r = \Delta(\Delta^{k-1} f_r), \quad E^k f_r = f_{r+k}.$$

Theorem 2: The polynomial $x \mapsto p_{n,m}^{\lambda}(x)$ is a particular solution of the following m-order differential equation

$$y^{(m)} + \sum_{s=0}^{m} a_s x^s y^{(s)} = 0, \tag{13}$$

where the coefficients a_s are given by

$$a_s = \frac{2^m}{s!m} \Delta^s f_0 \qquad (s = 0, 1, \ldots, m). \tag{14}$$

Proof: Let $n = pm + q$, where $p = [n/m]$ and $0 \leqslant q \leqslant m - 1$. By differentiating (7), we find

$$x^s D^s p_{n,m}^{\lambda}(x) = \sum_{k=0}^{\left[\frac{n-s}{m}\right]} (-1)^k \frac{(\lambda)_{n-(m-1)k}}{k!(n - mk - s)!}(2x)^{n-mk}$$

and

$$D^m p_{n,m}^{\lambda}(x) = \sum_{k=0}^{p-1} (-1)^k \frac{(\lambda)_{n-(m-1)k}\, 2^m}{k!(n - m(k + 1))!}(2x)^{n-m(k+1)},$$

where $\left[\dfrac{n - s}{m}\right] = p$ when $s \leqslant q$, or $= p - 1$ when $s > q$.

If we substitute these expressions in the differential equation (13) and compare the corresponding coefficients, we obtain the following relations:

$$\sum_{s=0}^{m} \binom{n - mk}{s} s!a_s = 2^m k(\lambda + n - (m - 1)k)_{m-1} \qquad (k = 0, 1, \ldots, p - 1) \tag{15}$$

and

$$\sum_{s=0}^{q} \binom{n - mp}{s} s!a_s = 2^m p(\lambda + n - (m - 1)p)_{m-1}.$$

First, we consider the second equality, i.e.,

$$\sum_{s=0}^{q} \binom{q}{s} \frac{2^m}{m} \Delta^s f_0 = 2^m \frac{n - q}{m}\left(\lambda + q + \frac{n - q}{m}\right)_{m-1}.$$

This equality is correct, because it is equivalent to

$$(1 + \Delta)^q f_0 = E^q f_0 = f_q = f(q).$$

Equality (15) can be written in the form

$$\sum_{s=0}^{m} \binom{n - mk}{s} \Delta^s f_0 = f_{n-mk} \qquad (k = 0, 1, \ldots, p - 1). \tag{16}$$

Since $t \mapsto f(t)$ is a polynomial of degree m, the last equalities are correct; (16) is a forward-difference formula for f at the point $t = n - mk$.

Thus, the proof is completed.

From (14), we have

$$a_0 = \frac{2^m n}{m}\binom{n + m\lambda}{m}_{m-1} = \frac{2^m n}{m^m}\prod_{i=1}^{m-1}(n + m(\lambda + i - 1)),$$

$$a_1 = \frac{2^m}{m}\left\{(n - 1)\binom{\frac{n - 1 + m(\lambda + 1)}{m}}{}_{m-1} - n\left(\frac{n + m\lambda}{m}\right)_{m-1}\right\}, \quad \text{etc.}$$

Since

$$f(t) = -\left(\frac{m - 1}{m}\right)^{m-1} t^m + \text{terms of lower degree},$$

we find

$$a_m = -\frac{2^m}{m}\left(\frac{m - 1}{m}\right)^{m-1}.$$

For $m = 1, 2, 3$, we have the following differential equations:

$$(1 - 2x)y' + 2ny = 0,$$

$$(1 - x^2)y'' - (2\lambda + 1)xy' + n(n + 2\lambda)y = 0,$$

$$\begin{aligned}\left(1 - \frac{32}{27}x^3\right)y''' &- \frac{16}{9}(2\lambda + 3)x^2y'' \\ &- \frac{8}{27}(3n(n + 2\lambda + 1) - (3\lambda + 2)(3\lambda + 5))xy' \\ &+ \frac{8}{27}n(n + 3\lambda)(n + 3(\lambda + 1))y = 0.\end{aligned}$$

Note that the second equation is the Gegenbauer equation.

REFERENCES

1. L. Gegenbauer. "Zur Theorie der Functionen $C_n^\nu(x)$." *Osterreichische Akademie der Wissenschaften Mathematisch Naturwissen Schaftliche Klasse Denkscriften 48* (1884):293-316.
2. H. W. Gould. "Inverse Series Relations and Other Expansions Involving Humbert Polynomials." *Duke Math. J. 32*, no. 4 (1965):697-712.
3. A. F. Horadam. "Polynomials Associated with Chebyshev Polynomials of the First Kind." *The Fibonacci Quarterly* **15**, no. 3 (1977):255-257.
4. A. F. Horadam. "Chebyshev and Fermat Polynomials for Diagonal Functions." *The Fibonacci Quarterly 17*, no. 4 (1979):328-333.
5. A. F. Horadam. "Gegenbauer Polynomials Revisited." *The Fibonacci Quarterly 23*, no. 4 (1985):294-299, 307.
6. A. F. Horadam & S. Pethe. "Polynomials Associated with Gegenbauer Polynomials." *The Fibonacci Quarterly 19*, no. 5 (1981):393-398.
7. D. V. Jaiswal. "On Polynomials Related to Tchebichef Polynomials of the Second Kind." *The Fibonacci Quarterly 12*, no. 3 (1974):263-265.
8. P. Humbert. "Some Extensions of Pincherle's Polynomials." *Proc. Edinburgh Math. Soc. (1) 39* (1921):21-24.

◆◇◆◇◆