ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-610 *Proposed by L. Kuipers, Serre, Switzerland*

Prove that there are no positive integers r, s, t such that (F_r, F_s, F_t) is a Pythagorean triple (that is, such that $F_r^2 + F_s^2 = F_t^2$).

B-611 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{k=1}^{n} L_{4k+2}.$$

For which positive integers n is $S(n)$ an integral multiple of 3?

B-612 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$T(n) = \sum_{k=1}^{n} F_{4k+2}.$$

For which positive integers n is $T(n)$ an integral multiple of 7?

B-613 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Show that there exist integers a, b, and c such that

$$F_{n+p}^2 + F_{n-p}^2 = aF_n^2F_p^2 + b(-1)^pF_n^2 + c(-1)^nF_p^2.$$

B-614 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $L(n) = L_{n-2}L_{n-1}L_{n+1}L_{n+2}$ and $F(n) = F_{n-2}F_{n-1}F_{n+1}F_{n+2}$. Show that

$$L(n) \equiv F(n) \pmod 8$$

and express $[L(n) - F(n)]/8$ as a polynomial in F_n.

B-615 *Proposed by Michael Eisenstein, San Antonio, TX*

Let $C(n) = L_n$ and $a_n = C(C(n))$. For $n = 0, 1, \ldots,$ prove that

$$a_{n+3} = a_{n+2}a_{n+1} \pm a_n.$$

SOLUTIONS

Fibonacci Convolution

B-586 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Show that $5\sum_{k=0}^{n} F_{k+1}F_{n+1-k} = (n+1)F_{n+3} + (n+3)F_{n+1}$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

It is known that

$$5(F_1F_{t-1} + F_2F_{t-2} + \cdots + F_{t-2}F_2 + F_{t-1}F_1)$$
$$= (t-1)F_{t+1} + (t+1)F_{t-1}.$$

[For a proof of this result, see (1.12) on p. 118 of "Fibonacci Convolution Sequences" by V. E. Hoggatt, Jr., and Marjorie Bicknell-Johnson, which appears in the April 1977 issue of this journal.] Thus,

$$\begin{aligned} 5\sum_{k=0}^{n} F_{k+1}F_{n+1-k} &= 5(F_1F_{n+1} + F_2F_n + \cdots + F_nF_2 + F_{n+1}F_1) \\ &= [(n+2) - 1]F_{(n+2)+1} + [(n+2) + 1]F_{(n+2)-1} \\ &= (n+1)F_{n+3} + (n+3)F_{n+1}. \end{aligned}$$

Also solved by Demetris Antzoulakos, Paul S. Bruckman, Lászlő Cseh, Russell Euler, Piero Filipponi & Odoardo Brugia, Herta T. Freitag, George Koutsoukellis, L. Kuipers, Jia-Sheng Lee, Carl Libis, Sahib Singh, J. Suck, Nico Trutzenberg, Gregory Wulczyn, and the proposer.

D. E. for Fibonacci Generating Function

B-587 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Let $y = \sum_{n=0}^{\infty} F_n x^n/n!$ and $z = \sum_{n=0}^{\infty} L_n x^n/n!$.

Show that $y'' = y' + y$ and $z'' = z' + z$.

Solution by Alberto Facchini, Università di Udine, Udine, Italy

Since

$$y' = \sum_{n=0}^{\infty} F_{n+1}x^n/n!, \quad y'' = \sum_{n=0}^{\infty} F_{n+2}x^n/n! \text{ and } F_{n+2} = F_{n+1} + F_n,$$

the desired result follows. The proof for z is similar.

Also solved by Demetris Antzoulakos, Charles Ashbacher, Paul S. Bruckman, Gabriel B. Costa, Lászlő Cseh, Russell Euler, Piero Filipponi, L. Kuipers, J.-S. Lee, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Closed Form Exponential Generating Function

B-588 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Find the y and z of Problem B-587 in closed form.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

Let $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$. Then,

$$y = \sum_{n=0}^{\infty} F_n x^n/n! = \sum_{n=0}^{\infty} \frac{a^n - b^n}{\sqrt{5}}(x^n/n!) = \frac{1}{\sqrt{5}}(e^{ax} - e^{bx})$$

and

$$z = \sum_{n=0}^{\infty} L_n x^n/n! = \sum_{n=0}^{\infty} (a^n + b^n)(x^n/n!) = e^{ax} + e^{bx}.$$

Also solved by Demetris Antzoulakos, Paul S. Bruckman, Lászlő Cseh, Russell Euler, Alberto Facchini, Piero Filipponi, Jia-Sheng Lee, Carl Libis, H.-J. Seiffert, Sahib Singh, Lawrence Somer, and the proposer.

Periodic Decimal Expansions

B-589 *Proposed by Herta T. Freitag, Roanoke, VA*

The number $N = 0434782608695652173913$ has the property that the digits of KN are a permutation of the digits of N for $K = 1, 2, \ldots, m$. Determine the largest such m.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

The largest such m is 22.

N consists of the 22 digits in the period (in base 10) for 1/23. As is easily checked, $23N$ is the 22-digit numeral each of whose digits is a 9.

Dickson reports: "J. W. L. Glaisher . . . noted that if q is a prime such that the period for $1/q$ has $q - 1$ digits, the products of the period for $1/q$ by $1, 2, \ldots, q - 1$ have the same digits in the same cyclic order. This property, well known for $q = 7$, holds also for $q = 17, 19, 23, 29, 47, 59, 61, 97$, and for $q = 7^2$." [See Dickson, *History of the Theory of Numbers*, Vol. I, p. 171 (New York: Chelsea Publishing Company, 1966).]

Also solved by Charles Ashbacher, Paul S. Bruckman, Piero Filipponi, L. Kuipers, Marjorie Johnson, Jia-Sheng Lee, Sahib Singh, Nico Trutzenberg, and the proposer.

Leftmost Digit

B-590 *Proposed by Herta T. Frietag, Roanoke, VA*

Generalize on Problem B-589 and describe a method for predicting the leftmost digit of KN.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh

For the generalization, see the solution to B-589.

Let q be a prime number such that the period for $1/q$ has $q - 1$ digits. Also, let M consist of the $q - 1$ digits in the period (in base 10) for $1/q$. To predict the leftmost digit of KM, $K = 1, 2, \ldots, q - 1$, write the digits of M in increasing order with each digit appearing in the sequence S_M exactly as many times as it appears in M. Then the leftmost digit of KM is the K^{th} entry in the sequence S_M. This follows from the fact that the products KM have the same digits as M in the same cyclic order and increase as K increases.

Example: For N, $S_N = 0, 0, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 6, 6, 6, 7, 7, 8, 8, 9, 9$. Thus, the leftmost digit of $6N$ is 2 and the leftmost digit of $12N$ is 5.

Editorial Note: Paul S. Bruckman gave the formula $[10K/q]$ for the leftmost digit of KN.

Also solved by Paul S. Bruckman, Piero Filipponi, Marjorie Johnson, Jia-Sheng Lee, Sahib Singh, and the proposer.

Interval With No Zeros

B-591 *Proposed by Mihaly Bencze, Jud. Brasa, Romania*

Let $F(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ with each a_n in $\{0, 1\}$.

Prove that $f(x) \neq 0$ for all x in $-1/a < x < 1/a$, where $a = (1 + \sqrt{5})/2$.

Solution by H.-J. Seiffert, Berlin, Germany

If $0 \leqslant x$, then, of course, $F(x) > 0$. Now assume that $-1/a < x < 0$. Then

$$F(x) = 1 + \sum_{n=1}^{\infty} a_n x^n \geqslant 1 + \sum_{k=1}^{\infty} a_{2k-1} x^{2k-1}$$

$$\geqslant 1 + \sum_{k=1}^{\infty} x^{2k-1} > 1 - \sum_{k=1}^{\infty} (1/a)^{2k-1} = \frac{a^2 - a - 1}{a^2 - 1} = 0.$$

Also solved by Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, L. Kuipers, and the proposer.

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