THE LENGTH OF A THREE-NUMBER GAME

JOSEPH W. CREELY
31 Chatham Place, Vincentown, NJ 08088

(Submitted April 1986)

1. THE THREE-NUMBER PROBLEM

Let $B = (b_1, b_2, b_3)$ represent a column vector of three elements and define the operator D_3 on B as

$$D_3(b_1, b_2, b_3) = (|b_1 - b_3|, |b_1 - b_2|, |b_2 - b_3|).$$

Given any initial vector B_0, we obtain a sequence $\{B_n\}$ with $B_n = D_3B_{n-1}$. This sequence is called the "three-number game" because of its similarity to the four-number game studied by Webb [2].

Define $rB = \max(|b_1|, |b_2|, |b_3|)$. Then, $rB \geqslant rD_3B$ with equality only if D_3B is of the form B', where

$$B' \in [(b', b', 0), (0, b', b'), (b', 0, b')], \quad b' \geqslant 0.$$

Definition 1.1: The length of the sequence $\{B_n\}$, denoted $L(B)$, is the smallest n such that B_n takes the form B'.

The three-number problem is to determine $L(B)$ given B. Note that, if $b_1 = b_2 = b_3$, $B' = 0$ and $L(B) = 1$.

Definition 1.2: If $L(B) = L(C)$, B and C are said to be virtually equivalent, $B \simeq C$.

Let $C_0 = P_0B_0$, a vector in which the elements of B_0 are rearranged, then $C_i = P_iB_i$, $i = 1, 2, \ldots, n$, where P_i is some permutation matrix. Therefore, $C_0 \simeq B_0$ and

$$B_0 \simeq P_0B_0. \tag{1.1}$$

Definition 1.3: The vector B is said to be *proper* if $B = (a, b, 0) + cU$, where $a > b \geqslant 0$, c is arbitrary, and $U = (1, 1, 1)$.

Note that either $L(B) = 1$ or B is virtually equivalent to a proper vector. If B is proper, then

$$D_3B \simeq \begin{cases} (b, 2b - a, 0) + (a - b)U & \text{if } 2b \geqslant a > b > 0, \\ (a - b, a - 2b, 0) + bU & \text{if } a \geqslant 2b. \end{cases} \tag{1.2}$$

In either case, D_3B is virtually equivalent to a proper vector of the form $(a', b', 0) + c_1U$, where $c_1 = a' - b'$ and is independent of c.

If c' is arbitrary and B is proper, then

$$B + c'U \simeq B. \tag{1.3}$$

If k is an integer and B is proper, $D_3kB = |k|D_3B$; hence,

$$kB \simeq B. \tag{1.4}$$

The three-number problem can be solved, in general, by use of the above equations. If B is proper, it reduces to a solution of the two-number problem as shown below.

2. THE TWO-NUMBER PROBLEM

The two-number game has been studied by the author (see [1]). Let D_2 represent an operator defined on a vector $A = (a, b)$, $a \geqslant b > 0$, by

$$D_2A = \begin{cases} (b, a - b) & 2b \geqslant a, \\ (a - b, b) & a \geqslant 2b. \end{cases} \tag{2.1}$$

Definition 2.1: The complement of A is defined as $C(a, b) = (a, a - b)$. Then, $C = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ and if $a > b > 0$,

$$D_2CA = D_2A. \tag{2.2}$$

Given any initial vector A_0, we obtain a sequence $\{A_n\}$ with $A_n = D_2A_{n-1}$. This sequence is called the "two-number game."

Definition 2.2: The length of the sequence $\{A_n\}$, denoted $L_2(A)$ or $L_2(a, b)$ is the smallest n such that $A_n = (a', 0)$ for some integer $a' > 0$.

It follows that $L_2(n, 1) = n$ and that

$$L_2(a, b) = [a/b] + L_2(b, a \pmod b), \tag{2.3}$$

where $[x]$ represents the greatest integer in the number x.

The two-number problem has been solved for $a \geqslant b > 0$ as the result of repeated applications of this formula.

3. THE MAIN RESULT

Theorem 3.1: If $B = (a, b, 0) + cU$ is proper, then $L(B) = L_2(a, b)$.

Proof: Comparing equations (2.1) and (1.2), we see that

$$D_3B_0 \simeq (CD_2A_0, 0) + c_1U \quad \text{or}$$
$$B_1 \simeq (CA_1, 0) + c_1U,$$
$$B_2 \simeq (CA_2, 0) + c_2U, \text{ etc., where } c_i \text{ is an integer.}$$

For some n, $B_n = (b', b', 0)$, $c_n = 0$, $B_{n-1} \neq B_n$, and $L(B_0) = n$, but $B_n \simeq (CA_n, 0)$, so $A_n = (b', 0)$. Since $D_2(b', 0)$ does not exist, there is only one n such that $A_n = (b', 0)$. It follows that, if $B = (A, 0) + cU$ is proper, then

$$L(B) = L_2(A). \blacksquare$$

REFERENCES

1. J. W. Creely. "The Length of a Two-Number Game." *The Fibonacci Quarterly* *25*, no. 2 (1987):174-179.
2. W. A. Webb. "The Length of a Four-Number Game." *The Fibonacci Quarterly* *20*, no. 1 (1982):33-35.

◆◇◆◇◆