ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-616 *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, MA*

(a) Find the smallest positive integer a such that

$$L_n \equiv F_{n+a} \pmod 6 \text{ for } n = 0, 1, \ldots .$$

(b) Find the smallest positive integer b such that

$$L_n \equiv F_{5n+b} \pmod 5 \text{ for } n = 0, 1, \ldots .$$

B-617 *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, MA*

Let R be a rectangle each of whose vertices has Fibonacci numbers as its coordinates x and y. Prove that the sides of R must be parallel to the coordinate axes.

B-618 *Proposed by Herta T. Treitag, Roanoke, VA*

Let $S(n) = L_{2n+1} + L_{2n+3} + L_{2n+5} + \cdots + L_{4n-1}$. Prove that $S(n)$ is an integral multiple of 10 for all even positive integers n.

B-619 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T(n) = F_{2n+1} + F_{2n+3} + F_{2n+5} + \cdots + F_{4n-1}$. For which positive integers n is $T(n)$ an integral multiple of 10?

B-620 *Proposed by Philip L. Mana, Albuquerque, NM*

Prove that $F^n_{24k+3} + F^n_{24k+5} \equiv 2F^n_{24k+6}$ (mod 9) for all n and k in $N = \{0, 1, 2, \ldots\}$.

B-621 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $n = 2h - 1$ with h a positive integer. Also, let $K(n) = F_h L_{h-1}$. Find sufficient conditions on F_n to establish the congruence

$$F^{K(n)}_{n+1} \equiv 1 \pmod{F_n}.$$

SOLUTIONS

No Such Constants

B-592 *Proposed by Herta T. Freitag, Roanoke, VA*

Find all integers a and b, if any, such that $F_a L_b + F_{a-1} L_{b-1}$ is an integral multiple of 5.

Solution by J.-Z. Lee, Chinese Culture University and J.-S. Lee, National Taipei Business College, Taipei, Taiwan, R.O.C.

Since $F_a L_b + F_{a-1} L_{b-1} = L_{a+b-1}$ and $L_n \equiv [2, 1, 3, 4]$ (mod 5), i.e., $L_n \not\equiv 0$ (mod 5), $F_a L_b + F_{a-1} L_{b-1}$ is not an integral multiple of 5 (for all integers a and b).

Also solved by P. S. Bruckman, F. H. Cunliffe, P. Filipponi, G. Koutsoukellis, L. Kuipers, B. Prielipp, H.-J. Seiffert, S. Singh, L. Somer, G. Wulczyn, and the proposer.

Multiple of 1220

B-593 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A(n) = F_{n+1} L_n + F_n L_{n+1}$. Prove that $A(15n - 8)$ is an integral multiple of 1220 for all positive integers n.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

By Problem B-294 on p. 375 of the December 1975 issue of this journal,

$$F_n L_k + F_k L_n = 2F_{n+k}.$$

Thus, $A(n) = 2F_{2n+1}$, so

$$A(15n - 8) = 2F_{30n-15} = 2F_{15(2n-1)}.$$

Because 15 divides $15(2n - 1)$, $610 = F_{15}$ divides $F_{15(2n-1)}$. Thus, $2(610) = 1220$ divides $A(15n - 8)$.

Also solved by P. S. Bruckman, F. H. Cunliffe, P. Filipponi, G. Koutsoukellis, L. Kuipers, J.-Z. Lee & J.-S. Lee, H.-J. Seiffert, S. Singh, L. Somer, G. Wulczyn, and the proposer.

Congruence Mod 60

B-594 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A(n) = F_{n+1}L_n + F_nL_{n+1}$ and $B(n) = \sum_{j=1}^{n}\sum_{k=1}^{j} A(k)$.

Prove that $B(n) \equiv 0 \pmod{20}$ when $n \equiv 19$ or $29 \pmod{60}$.

Solution by Paul S. Bruckman, Fair Oaks, CA

Using the expression derived in the solution to B-593, we have:

$$B(n) = \sum_{j=1}^{n}\sum_{k=1}^{j} 2F_{2k+1} = 2\sum_{j=1}^{n}\sum_{k=1}^{j}(F_{2k+2} - F_{2k}) = 2\sum_{j=1}^{n}(F_{2j+2} - F_2)$$
$$= 2\sum_{j=2}^{n+1}F_{2j} - 2n = 2\sum_{j=2}^{n+1}(F_{2j+1} - F_{2j-1}) - 2n = 2(F_{2n+3} - F_3) - 2n,$$

or

$$B(n) = 2F_{2n+3} - (2n + 4). \tag{1}$$

Now $(F_n \pmod 4)_{n=1}^{\infty}$ and $(F_n \pmod 5)_{n=1}^{\infty}$ are periodic sequences of periods 6 and 20, respectively. Thus, $(F_n \pmod{20})_{n=1}^{\infty}$ has period equal to L.C.M.(6, 20) = 60, from which it follows that $(F_{2n+3} \pmod{20})_{n=1}^{\infty}$ has period 30, as well as the sequence $(2F_{2n+3} \pmod{20})_{n=1}^{\infty}$. Also, $((2n + 4) \pmod{20})^{\infty}$ has period 10, clearly. Therefore, $(B(n) \pmod{20})_{n=1}^{\infty} \equiv ((2F_{2n+3} - (2n + 4)) \pmod{20})_{n=1}^{\infty}$ has period 30. Inspecting the 30 possible values of this sequence, we find that

$$B(n) \equiv 0 \pmod{20} \text{ iff } n \equiv 0, 19, \text{ or } 29 \pmod{30}.$$

This is a stronger result than sought in the problem.

Also solved by P. Filipponi, L. Kuipers, J.-Z. Lee & J.-S. Lee, B. Prielipp, S. Singh, G. Wulczyn, and the proposer.

Convolution Congruence

B-595 *Proposed by Philip L. Mana, Albuquerque, NM*

Prove that $\sum_{k=0}^{n} k^3(n - k)^2 \equiv \binom{n+4}{6} + \binom{n+1}{6} \pmod 5$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

It is known that

$$\sum_{k=0}^{n} k^3(n - k)^2 = \binom{n+1}{6} + 5\binom{n+2}{6} + 5\binom{n+3}{6} + \binom{n+4}{6}.$$

(See p. 57 of "A Symmetric Substitute for Sterling Numbers" by A. P. Hillman, P. L. Mana, and C. T. McAbee in the February 1971 issue of this journal.) The desired result follows immediately.

Also solved by P. S. Bruckman, P. Filipponi, H. T. Freitag, L. Kuipers, J.-Z. Lee & J.-S. Lee, S. Singh, G. Wulczyn, and the proposer.

X, Y, Z Affair

B-596 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let

$$S(n, k, m) = \sum_{i=1}^{m} F_{ni+k}.$$

For positive integers a, m, and k, find an expression of the form XY/Z for $S(4a, k, m)$, where X, Y, and Z are Fibonacci or Lucas numbers.

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

Let $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$. Using the Binet form for Fibonacci numbers,

$$S(n, k, m) = \frac{1}{\alpha - \beta}\left[\sum_{i=1}^{m} \alpha^{ni+k} - \sum_{i=1}^{m} \beta^{ni+k}\right]$$

$$= \frac{F_{(m+1)n+k} - F_{n+k} - (\alpha\beta)^n\{F_{mn+k} - F_k\}}{L_n - 1 - (\alpha\beta)^n}.$$

Thus,

$$S(4a, k, m) = \frac{F_{4a(m+1)+k} - F_{4a+k} - \{F_{4am+k} - F_k\}}{L_{4a} - 2}$$

$$= \frac{F_{2am}L_{2am+4a+k} - F_{2am}L_{2am+k}}{5F_{2a}^2}$$ by I_{16} and I_{24} of Hoggatt's *Fibonacci and Lucas Numbers*

$$= \frac{F_{2am}(5F_{2a} \cdot F_{2am+2a+k})}{5F_{2a}^2} = \frac{F_{2am} \cdot F_{2am+2a+k}}{F_{2a}} = \frac{XY}{Z},$$

where X, Y, and Z are all Fibonacci numbers.

Also solved by P. S. Bruckman, H. T. Freitag, J.-Z. Lee & J.-S. Lee, H.-J. Seiffert, G. Wulczyn, and the proposer.

More X, Y, Z Relations

B-597 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Do as in Problem B-596 for $S(4a + 2, k, 2b)$ and for $S(4a + 2, k, 2b - 1)$, where a and b are positive integers.

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

Using the result in B-596, we obtain:

Case I

$$S(4a + 2, k, 2b) = \frac{F_{2(2a+1)(2b+1)+k} - F_{2(2a+1)+k} - \{F_{4b(2a+1)+k} - F_k\}}{L_{4a+2} - 2}$$

$$= \frac{(F_{2(2a+1)(2b+1)+k} - F_{4b(2a+1)+k}) - \{F_{2(2a+1)+k} - F_k\}}{L_{2a+1}^2}$$

$$= \frac{F_{(2a+1)(4b+1)+k}L_{2a+1} - F_{(2a+1+k)}L_{2a+1}}{L^2_{2a+1}}$$

$$= \frac{F_{(2a+1)(4b+1)+k} - F_{2a+1+k}}{L_{2a+1}}$$

$$= \frac{F_{2(2a+1)b}L_{(2a+1)(2b+1)+k}}{L_{2a+1}},$$

by using I_{18}, I_{23}, and I_{24} in Hoggatt's *Fibonacci and Lucas Numbers.*

Case 2

$$S(4a+2, k, 2b-1) = \frac{F_{4(2a+1)b+k} - F_{2(2a+1)+k} - \{F_{2(2a+1)(2b-1)+k} - F_k\}}{L_{4a+2} - 2}$$

$$= \frac{F_{4(2a+1)b+k} - F_{2(2a+1)(2b-1)+k} - \{F_{2(2a+1)+k} - F_k\}}{L^2_{2a+1}}$$

$$= \frac{F_{(2a+1)(4b-1)+k}L_{2a+1} - F_{2a+1+k}L_{2a+1}}{L^2_{2a+1}}$$

$$= \frac{F_{(2a+1)(4b-1)+k} - F_{2a+1+k}}{L_{2a+1}}$$

$$= \frac{F_{2(2a+1)b+k}L_{(2a+1)(2b-1)}}{L_{2a+1}}.$$

Also solved by P. S. Bruckman, H. T. Freitag, L. Kuipers, J.-Z. Lee & J.-S. Lee, H.-J. Seiffert, G. Wulczyn, and the proposer.

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