ON THE L^p-DISCREPANCY OF CERTAIN SEQUENCES

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1. INTRODUCTION

Let (x_n), $n = 1, 2, \ldots$ be a sequence of real numbers contained in $[0, 1)$. Let $A([0, x); N)$ be the number of x_n, $1 \leqslant n \leqslant N$, that lie in the subinterval $[0, x)$ of the unitinterval. The number

$$D_N^{(p)} = \left(\int_0^1 \left|\frac{A([0, x); N)}{N} - x\right|^p dx\right)^{1/p}, \ldots, \tag{1}$$

where $1 \leqslant p < \infty$, is called the L^p-discrepancy of the given sequence ([2], p. 97).

As is well known, the notion of discrepancy is at the center of most theories in the area of uniform distribution (and other types of distributions as well) and quantitative aspects of certain limit passages are expressed by estimates of the discrepancy.

The following relation was given by Koksma [1] and by Niederreiter [3]:

$$(D_N^{(2)})^2 = \frac{1}{12N^2} + \frac{1}{N}\sum_{n=1}^{N}(x_n - s_n)^2, \ldots, \tag{2}$$

where $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant 1$ and $s_n - (2n - 1)/2N$.

In the following, we show (2) (for the sake of completeness) and consider the case $p = 4$. The proofs are given by elementary methods. Some sum formulas are established and only integration results are used.

2. THE CASE $p = 2$

To prove (2), the following lemma is useful.

Lemma 1: $\sum_{n=1}^{N}\sum_{m=1}^{N}\max(x_n, x_m) = \sum_{n=1}^{N}(2n - 1)x_n$, where $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1$.

Proof: $\sum_{n=1}^{N}\sum_{m=1}^{N}\max(x_n, x_m) = \sum_{n=1}^{N}\left(\sum_{m=1}^{N}\max(x_n, x_m)\right) = \sum_{n=1}^{N}(2n - 1)x_n$,

since for any n there are n values of m satisfying $m \leqslant n$ taking care of the $2n$ pairs $(x_n, x_1), \ldots, (x_n, x_n), (x_1, x_n), \ldots, (x_n, x_n)$. But (x_n, x_n) is counted twice, so for any n there are $2n - 1$ values of x_n.

Let $c(t, x_n) = \begin{cases} 0 (x_n \geqslant t) \\ 1 (x_n < t) \end{cases}$, $0 \leqslant t \leqslant 1$. Then

$$\int_0^1 \left(\sum_{n=1}^N c(t, x_n) \right)^2 dt = \int_0^1 \sum_{n=1}^N \sum_{m=1}^N c(t, x_n) c(t, x_m) dt$$
$$= \sum_{n=1}^N \sum_{m=1}^N \int_0^1 c(t, \max(x_n, x_m)) dt.$$

Now we show (2). We have:

$$(ND_N^{(2)})^2 = \int_0^1 \left| \sum_{n=1}^N c(t, x_n) - Nt \right|^2 dt$$
$$= \int_0^1 \left(\sum_{n=1}^N c(t, x_n) \right)^2 dt - 2N \int_0^1 t \sum_{n=1}^N c(t, x_n) dt + N^2 \int_0^1 t^2 \, dt$$
$$= \sum_{n=1}^N \sum_{m=1}^N \int_0^1 c(t, \max(x_n, x_m)) dt - 2N \int_0^1 t \sum_{n=1}^N c(t, x_n) dt + \frac{1}{3} N^2$$
$$= \sum_{n=1}^N \sum_{m=1}^N (1 - \max(x_n, x_m)) - 2N \sum_{n=1}^N \frac{1}{2}(1 - x_n^2) + \frac{1}{3} N^2$$
$$= N^2 - \sum_{n=1}^N \sum_{m=1}^N \max(x_n, x_m) - N\left(N - \sum_{n=1}^N x_n^2\right) + \frac{1}{3} N^2$$
$$= \frac{1}{3} N^2 - \sum_{n=1}^N (2n - 1) x_n + N \sum_{n=1}^N x_n^2 \quad \text{(by Lemma 1)}.$$

Hence,

$$(D_N^{(2)})^2 = \frac{1}{3} + \frac{1}{N} \sum_{n=1}^N \left(x_n^2 - \frac{2n - 1}{N} x_n \right) = \frac{1}{3} + \frac{1}{N} \sum_{n=1}^N (x_n^2 - 2s_n x_n)$$
$$= \frac{1}{3} + \frac{1}{N} \sum_{n=1}^N (x_n - s_n)^2 - \frac{1}{N} \sum_{n=1}^N s_n^2.$$

Since

$$\sum_{n=1}^N s_n^2 = \sum_{n=1}^N \left(\frac{2n - 1}{2N} \right)^2 = \frac{1}{4N^2} \sum_{n=1}^N (4n^2 - 4n + 1) = \frac{1}{4N^2} \left(\frac{4N^3}{3} - \frac{N}{3} \right),$$

we have

$$(D_N^{(2)})^2 = \frac{1}{N} \sum_{n=1}^N (x_n - s_n)^2 + \frac{1}{12N^2},$$

and this proves (2).

Corollary 1: $D_N^{(2)} \geqslant \dfrac{1}{2N\sqrt{3}}$; $D_N^{(2)} = \dfrac{1}{2N\sqrt{3}}$ iff $x_n = s_n$ $(n = 1, 2, \ldots, N)$.

Corollary 2: $D_N^{(2)} \leqslant \dfrac{1}{\sqrt{3}}$ if $x_n \leqslant \dfrac{2n - 1}{N}$ $(n = 1, 2, \ldots, N)$.

Proof: $(D_N^{(2)})^2 = \frac{1}{3} + \frac{1}{N}\sum_{n=1}^{N} x_n\left(x_n - \frac{2n-1}{N}\right)$; hence,

$(D_N^{(2)}) \leqslant \frac{1}{3}$ if $x_n \leqslant \frac{2n-1}{N}$ $(n = 1, 2, \ldots, N)$.

3. THE CASE $p = 4$

We shall use the following lemma.

Lemma 2: Let $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1$. Then,

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max{}^2(x_n, x_m, x_\ell) = \sum_{n=1}^{N}(3n^2 - 3n + 1)x_n^2.$$

Proof: $\displaystyle\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max{}^2(x_n, x_m, x_\ell) = \sum_{n=1}^{N}\sum_{m=1}^{N}(2m-1)\max{}^2(x_n, x_m)$

$$= 2\sum_{n=1}^{N}\sum_{m=1}^{N} m \max{}^2(x_n, m) - \sum_{n=1}^{N}\sum_{m=1}^{N}\max{}^2(x_n, x_m).$$

Now, $\displaystyle\sum_{n=1}^{N}\sum_{m=1}^{N}\max{}^2(x_n, x_m) = \sum_{n=1}^{N}(2n-1)x_n^2$ (by Lemma 1).

$$\sum_{n=1}^{N}\sum_{m=1}^{N} m \max{}^2(x_n, x_m) = \sum_{n=1}^{N}[2\{1 + 2 + \cdots + (n-1)\} + n]x_n^2$$

$$= \frac{1}{2}\sum_{n=1}^{N} n(3n-1)x_n^2.$$

Hence, we have

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max{}^2(x_n, x_m, x_\ell) = \sum_{n=1}^{N}(3n^2 - 3n + 1)x_n^2.$$

Corollary of Lemma 2:

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max(x_n, x_m, x_\ell) = \sum_{n=1}^{N}(3n^2 - 3n + 1)x_n.$$

Lemma 3: Let $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1$. Then,

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\max(x_n, x_m, x_\ell, x_u) = \sum_{n=1}^{N}(4n^3 - 6n^2 + 4n - 1)x_n.$$

Proof: First we consider

$$\sum_{n=1}^{N}\sum_{m=1}^{N} m^2 \max(x_n, x_m) = \sum_{n=1}^{N}\left(\sum_{m=1}^{N} m^2 \max(x_n, x_m)\right).$$

Keeping n fixed, we have to take the pairs $(x_1, x_n), \ldots, (x_n, x_n), (x_n, x_1), \ldots, (x_n, x_{n-1})$ into consideration; the maximums of the first set must be multiplied by n^2, those of the right set by $1^2, 2^2, \ldots, (n-1)^2$, respectively. Hence,

$$\sum_{n=1}^{N}\left(\sum_{m=1}^{N} m^2 \max(x_n, x_m)\right) = \sum_{n=1}^{N}\left\{\frac{1}{6}(n-1)n(2n-1) + n^3\right\}x_n.$$

So we have

$$\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\max(x_n, x_m, x_\ell, x_u)$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N}(3m^2 - 3m + 1)\max(x_n, x_m) \quad \text{(by the Corollary of Lemma 2)}$$

$$= 3\sum_{n=1}^{N}\sum_{m=1}^{N} m^2 \max(x_n, x_m) - 3\sum_{n=1}^{N}\sum_{m=1}^{N} m \max(x_n, x_m) + \sum_{n=1}^{N}\sum_{m=1}^{N}\max(x_n, x_m)$$

$$= 3\sum_{n=1}^{N}\left\{\frac{1}{6}n(n-1)(2n-1) + n^3\right\}x_n - 3\sum_{n=1}^{N}\frac{1}{2}n(3n-1)x_n + \sum_{n=1}^{N}(2n-1)x_n$$

$$= \sum_{n=1}^{N}(4n^3 - 6n^2 + 4n - 1)x_n.$$

Lemma 4: $1^3 + 2^3 + \cdots + N^3 = \frac{1}{4}N^2(N+1)^2,$

$$1^4 + 2^4 + \cdots + N^4 = \frac{1}{30}N(N+1)(2N+1)(3N^2 + 3N - 1).$$

Theorem: Let $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1$. Then,

$$(D_N^{(4)})^4 = \frac{1}{N}\sum_{n=1}^{N}\left\{(x_n - s_n)^4 + \frac{1}{2N^2}(x_n - s_n)^2\right\} + \frac{1}{80N^4}, \text{ where } s_n = \frac{2n-1}{2N}.$$

Proof: First, we have

$$-4N^3\sum_{n=1}^{N}\int_0^1 c(t, x_n)t^3\,dt = -4N^3\int_0^1\sum_{n=1}^{N}c(t, x_n)t^3\,dt$$

$$= -4N^3\sum_{n=1}^{N}\int_{x_n}^1 t^3\,dt = -N^3\sum_{n=1}^{N}(1 - x_n^4)$$

$$= -N^4 + N^3\sum_{n=1}^{N}x_n^4.$$

Second, we have

$$6N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\int_0^1 t^2 c(t, \max(x_n, x_m))dt = 6N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{1}{3}t^3\Big]_{\max(x_n, x_m)}^1$$

$$= 6N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\left(\frac{1}{3} - \frac{1}{3}(\max(x_n, x_m))^3\right) = 6N^2\left(\frac{N^2}{3} - \frac{1}{3}\sum_{n=1}^{N}\sum_{m=1}^{N}(\max(x_n, x_m))^3\right)$$

$$= 2N^4 - 2N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\max{}^3(x_n, x_m).$$

Hence,

$$(ND_N^{(4)})^4 = \int_0^1\left\{\sum_{n=1}^{N}c(t, x_n) - Nt\right\}^4 dt$$

(continued)

$$= \int_0^1 \left(\sum_{n=1}^{N} c(t, x_n)\right)^4 dt - 4N\int_0^1 t\left(\sum_{n=1}^{N} c(t, x_n)\right)^3 dt - 4N^3\int_0^1 t^3 \sum_{n=1}^{N} c(t, x_n)dt$$

$$+ 6N^2\int_0^1 t^2\left(\sum_{n=1}^{N} c(t, x_n)\right)^2 dt + N^4\int_0^1 t^4\, dt$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\int_0^1 c(t, \max(x_n, x_m, x_\ell, x_u)dt - 4N^3\sum_{n=1}^{N}\int_0^1 c(t, x_n)t^3\, dt$$

$$+ 6N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\int_0^1 c(t, \max(x_n, x_m))t^2\, dt$$

$$- 4N\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\int_0^1 tc(t, \max(x_n, x_m, x_\ell))dt + \frac{1}{5}N^4$$

$$= \frac{1}{5}N^4 - N^4 + N^3\sum_{n=1}^{N} x_n^4 + 2N^4 - 2N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\max{}^3(x_n, x_m)$$

$$- 2N^4 + 2N\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max{}^2(x_n, x_m, x_\ell)$$

$$+ N^4 - \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\max(x_n, x_m, x_\ell, x_u)$$

$$= \frac{1}{5}N^4 + N^3\sum_{n=1}^{N} x_n^4 - 2N^2\sum_{n=1}^{N}\sum_{m=1}^{N}\max{}^3(x_n, x_m)$$

$$+ 2N\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\max{}^2(x_n, x_m, x_\ell)$$

$$- \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\ell=1}^{N}\sum_{u=1}^{N}\max(x_n, x_m, x_\ell, x_u)$$

$$= \frac{1}{5}N^4 + N^3\sum_{n=1}^{N} x_n^4 - 2N^2\sum_{n=1}^{N}(2n-1)x_n^3$$

$$+ 2N\sum_{n=1}^{N}(3n^2 - 3n + 1)x_n^2 - \sum_{n=1}^{N}(4n^3 - 6n^2 + 4n - 1)x_n.$$

Hence,

$$(D_N^{(4)})^4 = \frac{1}{5} + \frac{1}{N}\sum_{n=1}^{N}\left(x_n^4 - \frac{2(2n-1)}{N}x_n^3 + \frac{2(3n^2 - 3n + 1)}{N^2}x_n^2 - \frac{4n^3 - 6n^2 + 4n - 1}{N^3}x_n\right)$$

$$= \frac{1}{5} + \frac{1}{N}\sum_{n=1}^{N}\left((x_n - s_n)^4 + \frac{1}{2N^2}(x_n - s_n)^2\right) - \frac{1}{N}\sum_{n=1}^{N}\left(s_n^4 + \frac{s_n^2}{2N^2}\right).$$

Now, $$\sum_{n=1}^{N}\left(s_n^4 + \frac{s_n^2}{2N^2}\right) = \sum_{n=1}^{N}\left(\left(\frac{2n-1}{2N}\right)^4 + \frac{(2n-1)^2}{8N^4}\right)$$

$$= \frac{1}{16N^4}\sum_{n=1}^{N}(16n^4 - 32n^3 + 32n^2 - 16n + 3)$$

$$= \frac{1}{16N^4}\left(16\sum_{n=1}^{N} n^4 - 32\sum_{n=1}^{N} n^3 + 32\sum_{n=1}^{N} n^2 - 16\sum_{n=1}^{N} n + \sum_{n=1}^{N} 3\right)$$

$$= \frac{1}{16N}\left(\frac{16}{5} N^5 - \frac{1}{5} N\right).$$

Finally,

$$(D_N^{(4)})^4 = \frac{1}{5} + \frac{1}{N}\sum_{n=1}^{N}\left((x_n - s_n)^4 + \frac{1}{2N^2}(x_n - s_n)^2\right) - \frac{1}{N}\cdot\frac{1}{16N^4}\left(\frac{16N^5}{5} - \frac{N}{5}\right)$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left((x_n - s_n)^4 + \frac{1}{2N^2}(x_n - s_n)^2\right) + \frac{1}{80N^4}.$$

Corollary 1: $D_N^{(4)} \geqslant \dfrac{1}{2N\sqrt[4]{5}}$; $D_N^{(4)} = \dfrac{1}{2N\sqrt[4]{5}}$ iff $x_n = s_n$ $(n = 1, 2, \ldots, N)$.

Corollary 2: $D_N^{(4)} \leqslant \dfrac{1}{\sqrt[4]{5}}$ if $x_n \leqslant \dfrac{2n-1}{N}$.

Proof: $$(D_4^{(4)})^4 = \frac{1}{5} + \frac{1}{N}\sum_{n=1}^{N}\left\{x_n\left(x_n - \frac{2n-1}{N}\right)\left(x_n^2 - \frac{2n-1}{N}x_n + \frac{2n^2-2n+1}{N^2}\right)\right\}.$$

Now, $$x_n - \frac{2n-1}{N}x_n + \frac{2n^2-2n+1}{N^2} = \left(x_n - \frac{2n-1}{2N}\right)^2 + \frac{4n^2-4n+3}{4N^2} > 0.$$

Hence, $D_4^{(4)} \leqslant \dfrac{1}{\sqrt[4]{5}}$.

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