ON CERTAIN DIVISIBILITY SEQUENCES

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In [1], Marshall Hall defined U_n to be a *divisibility sequence* if $U_m | U_n$ whenever $m | n$. If we let $U_n = A^n(c_0 + c_1 n)$ for integers A, c_0, and c_1, then a corollary to the theorem in [2] is that U_n is a divisibility sequence if and only if exactly one of the coefficients c_0 or c_1 equals 0. The purpose of this paper is to establish a similar result for $U_n = A^n(c_0 + c_1 n + c_2 n^2)$.

Theorem: Let $U_n = A^n(c_0 + c_1 n + c_2 n^2)$ for integers A, c_0, c_1, and c_2. U_n is a divisibility sequence if and only if exactly two of the coefficients c_0, c_1, and c_2 are 0.

Proof: It is easy to see that, if exactly two of the coefficients c_0, c_1, and c_2 are 0, then U_n is a divisibility sequence. Consequently, in what follows, we assume that $A^m(c_0 + c_1 m + c_2 m^2) | A^n(c_0 + c_1 n + c_2 n^2)$ if $m | n$, and, without loss of generality, that $A > 0$.

Case 1: $c_0 = 0$

Assume $c_1 \neq 0$, for, otherwise, we have $c_0 = c_1 = 0$ and $c_2 m^2 A | c_2 n^2 A$ if $m | n$, and we are finished. Replace m by $c_1 mA$, n by $c_1 nA$, and let $e = c_1 A(n - m)$. Then we have $(c_1^2 mA + c_2 c_1^2 m^2 A^2) | A^e (c_1^2 nA + c_2 c_1^2 n^2 A^2)$ if $m | n$. Consequently,

$$(m + c_2 m^2 A) | A^e (n + c_2 n^2 A) \text{ if } m | n.$$

In particular,

$$(1 + c_2 A) | A^e (n + c_2 n^2 A).$$

If $e \leqslant 0$, then $(1 + c_2 A) | (n + c_2 n^2 A)$ is immediate, while if $e > 0$, since

$$\gcd(1 + c_2 A, A^e) = 1,$$

we also have $(1 + c_2 A) | (n + c_2 n^2 A)$.

Set $n = 2$. $(1 + c_2 A) | (2 + 4c_2 A)$. Since $2 + 4c_2 A = 2(1 + c_2 A) + 2c_2 A$, we have $(1 + c_2 A) | 2c_2 A$, which implies that $(1 + c_2 A) | 2$; hence, $1 + c_2 A = \pm 1$ or ± 2.

$1 + c_2 A = 1 \Rightarrow c_2 = 0$, and we are finished.

$1 + c_2 A = -1 \Rightarrow (m - 2m^2) | (n - 2n^2)$ if $m | n$ and m is odd, which is false for $m = 3$, $n = 6$.

$1 + c_2A = 2 \Rightarrow c_2A = 1 \Rightarrow (m + m^2) | (n + n^2)$ if $m|n$, which is false for $m = 2$, $n = 4$.

$1 + c_2A = -2 \Rightarrow A = 1$ or $A = 3 \Rightarrow (m - 3m^2) | A^e(n - 3n^2)$ if $m|n$, which is false for $m = 5$, $n = 10$.

Case 2: $c_0 \neq 0$

Replace m by c_0mA, n by c_0nA, and let $e = c_0A(n - m)$. This gives

$$(c_0 + c_0c_1mA + c_2c_0^2m^2A^2) | A^e(c_0 + c_0c_1nA + c_2c_0^2n^2A^2),$$

which implies that

$$(1 + c_1mA + c_2c_0m^2A^2) | A^e(1 + c_1nA + c_2c_0n^2A^2).$$

As in Case 1, this leads to

$$(1 + c_1mA + c_2c_0m^2A^2) | (1 + c_1nA + c_2c_0n^2A^2) \text{ if } m|n.$$

Select $m = 1$, $n = 1 + c_1A + c_2c_0A^2$. Then $(1 + c_1A + c_2c_0A^2) | 1$, i.e.,

$$1 + c_1A + c_2c_0A^2 = \pm 1.$$

Case a: $1 + c_1A + c_2c_0A^2 = 1$

$1 + c_1A + c_2c_0A^2 = 1 \Rightarrow A(c_1 + c_2c_0A) = 0 \Rightarrow c_2c_0A = -c_1$. Thus,

$$(1 + c_1mA - c_1m^2A) | (1 + c_1nA - c_1n^2A) \text{ if } m|n.$$

Set $n = 2m$. $(1 + c_1mA - c_1m^2A) | (1 + 2c_1mA - 4c_1m^2A)$ if $m|n$, or

$$(1 + c_1mA - c_1m^2A) | (1 + c_1mA - c_1m^2A + (c_1mA - 3c_1m^2A)).$$

Hence,

$$(1 + c_1mA - c_1m^2A) | 2(c_1mA - 3c_1m^2A). \quad (1)$$

Set $n = 3m$. In a similar manner to the above, we get

$$(1 + c_1mA - c_1m^2A) | (2c_1mA - 8c_1m^2A). \quad (2)$$

Together, (1) and (2) imply that $(1 + c_1mA - c_1m^2A) | (2c_1m^2A)$.

Set $m = 2$. We obtain $(1 - 2c_1A) | 8c_1A$. But $8c_1A = 4 - 4(1 - 2c_1A)$, so that $(1 - 2c_1A) | 4$, i.e., $1 - 2c_1A = \pm 1$.

$1 - 2c_1A = 1 \Rightarrow c_1 = 0$. Since $c_2c_0A = -c_1$, either $c_0 = 0$ or $c_2 = 0$, and we are finished.

$1 - 2c_1A = -1 \Rightarrow c_2A = 1 \Rightarrow (1 + m - m^2) | (1 + n - n^2)$ if $m|n$, which is false for $m = 3$, $n = 6$.

Case b: $1 + c_1A + c_2c_0A^2 = -1$

$1 + c_1A + c_2c_0A^2 = -1 \Rightarrow A(c_1 + c_2c_0A) = -2 \Rightarrow A = 1$ or 2.

Case i: $A = 1,\ c_1 + c_2c_0 = -2$

If $A = 1$, then

$$(1 + c_1m + c_2c_0m^2) \mid (1 + c_1n + c_2c_0n^2) \text{ if } m \mid n.$$

Let $m = 2$ and replace n by $2n$. Then

$$(1 + 2c_1 + 4c_2c_0) \mid (1 + 2c_1n + 4c_2c_0n^2).$$

Since $c_1 + c_2c_0 = -2$, we have

$$(2c_2c_0 - 3) \mid (1 + 2c_1n + 4c_2c_0n^2).$$

Let $n = 2c_2c_0 - 3$. Then $(2c_2c_0 - 3) \mid 1 \Rightarrow 2c_2c_0 - 3 = \pm 1$.

$2c_2c_0 - 3 = 1 \Rightarrow c_2c_0 = 4 \Rightarrow c_1 = -4 \Rightarrow (1 - 4m + 2m^2) \mid (1 - 4n + 2n^2)$ if $m \mid n$, which is false for $m = 4$, $n = 8$.

$2c_2c_0 - 3 = -1 \Rightarrow c_2c_0 = 1 \Rightarrow c_1 = -3 \Rightarrow (1 - 3m + m^2) \mid (1 - 3n + n^2)$ if $m \mid n$, which is false for $m = 4$, $n = 8$.

Case ii: $A = 2,\ c_1 + 2c_2c_0 = -1$

If $A = 2$, then

$$(1 + 2c_1m + 4c_2c_0m^2) \mid (1 + 2c_1n + 4c_2c_0n^2) \text{ if } m \mid n.$$

Let $m = 2$, and replace n by $2n$. Consequently,

$$(1 + 4c_1 + 16c_2c_0) \mid (1 + 4c_1n + 16c_2c_0n^2).$$

Since $c_1 + 2c_2c_0 = -1$, we have

$$(8c_2c_0 - 3) \mid (1 + 4c_1n + 16c_2c_0n^2).$$

Let $n = 8c_2c_0 - 3$. Then $(8c_2c_0 - 3) \mid 1$, which is impossible.

Remark: It is reasonable to conjecture that

$$U_n = A^n \sum_{i=0}^{k} c_i n^i$$

is a divisibility sequence if and only if exactly k of the c_i's are 0. It appears that this general case cannot be proved using the methods in this paper.

REFERENCES

1. Marshall Hall. "Divisibility Sequences of 3rd Order." *Amer. J. Math.* 58 (1936):577-584.
2. R. B. McNeill. "A Note on Divisibility Sequences." *The Fibonacci Quarterly* (to appear).

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