ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$ 

PROBLEMS PROPOSED IN THIS ISSUE

B-662 Proposed by Philip L. Mana, Albuquerque, NM

For fixed $n$, find all $m$ such that $L_n F_m - F_{m+n} = (-1)^n$.

B-623 Proposed by Herta T. Freitag, Roanoke, VA

Let

$$S(n) = \sum_{k=1}^{2n-1} L_{n+k} L_k.$$ 

Prove that $S(n)$ is an integral multiple of $L_n$ for all positive integers $n$.

B-624 Proposed by Herta T. Freitag, Roanoke, VA

Let

$$T_n = \sum_{i=1}^{n} L_{2(n+i)-1}.$$ 

For every positive integer $n$, prove that either $P_n | T_n$ or $L_n | T_n$.

B-625 Proposed by H.-J. Seiffert, Berlin, Germany

Let $P_0, P_1, \ldots$ be the Pell numbers defined by

$$P_0 = 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2.$$ 

Let $G_n = F_n P_n$ and $H_n = L_n P_n$. Show that $(G_n)$ and $(H_n)$ satisfy

$$K_{n+4} - 2K_{n+3} - 7K_{n+2} - 2K_{n+1} + K_n = 0.$$ 

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Let \( G_n \) and \( H_n \) be as in B-625. Express the generating functions

\[
G(z) = \sum_{n=0}^{\infty} G_n z^n \text{ and } H(z) = \sum_{n=0}^{\infty} H_n z^n
\]

as rational functions of \( z \).

Proposed by H.-J. Seiffert, Berlin, Germany

B-626

Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Let

\[
C_{n,k} = \left( \frac{F_n^3 + F_{n+1}^3 + \cdots + F_{n+k-1}^3}{k} \right)
\]

Find the smallest \( k \) in \( \{2, 3, 4, \ldots \} \) such that \( C_{n,k} \) is an integer for every \( n \) in \( \{0, 1, 2, \ldots \} \).

Solutions

2 Problems on Pythagorean Triples

B-598 Proposed by Herta T. Freitag, Roanoke, VA

For which positive integers \( n \) is \((2L_n, L_{2n} - 3, L_{2n} - 1)\) a Pythagorean triple? For which of these \( n \)'s is the triple primitive?

B-599 Proposed by Herta T. Freitag, Roanoke, VA

Do B-598 with the triple now \((2L_n, L_{2n} + 1, L_{2n} + 3)\).

Solutions by Thomas M. Green, Contra Costa College, San Pablo, CA

It is known that \( L_{2n} = L_n^2 + 2(-1)^{n+1} \).

For \( n \) odd, we have \( L_{2n} = L_n^2 + 2 \) and the triple

\((2L_n, L_{2n} - 3, L_{2n} - 1) = (2L_n, L_n^2 - 1, L_n^2 + 1)\)

which is a Pythagorean triple. Furthermore, a Pythagorean triple of the type \((2m, m^2 - 1, m^2 + 1)\) is primitive if \( m \) is even. Thus, if \( L_n = m \), an even number, then \((2L_n, L_n^2 - 1, L_n^2 + 1)\) is primitive. But, if \( n \) is odd, \( L_n \) is even only when \( n \) is an odd multiple of three.

Similarly, for \( n \) even (B-599), the triple

\((2L_n, L_{2n} + 1, L_{2n} + 3) = (2L_n, L_n^2 - 1, L_n^2 + 1)\)

is Pythagorean and will be primitive if \( L_n \) is even. In this case, however, if \( n \) is even, \( L_n \) is even only when \( n \) is an even multiple of three.


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Fibonacci Multiples of 121,160

B-600 Proposed by Philip L. Mana, Albuquerque, NM

Let \( n \) be any positive integer and \( m = n^{13} - n \). Prove that \( F_m \) is an integral multiple of 30290.

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

We prove a more general result, namely: \( F_m \) is an integral multiple of 121,160, where \( m = n^{13} - n \); \( n \) being a positive integer.

We can express

\[
\begin{align*}
n^{13} - n &= (n^7 - n)(n^6 + 1) = (n^5 - n)(n^8 + n^4 + 1) \\
&= (n^3 - n)(n^{10} + n^8 + n^6 + n^4 + n^2 + 1).
\end{align*}
\]

By Fermat's theorem: \( n^p - n \equiv 0 \pmod{p} \), where \( p \) is prime and \( n \) is a positive integer.

Thus, we conclude that:

\( n^{13} - n \equiv 0 \pmod{13} \); \( n^{13} - n \equiv 0 \pmod{7} \); \( n^{13} - n \equiv 0 \pmod{5} \).

Since \( n^3 - n \) is a factor of \( n^{13} - n \) and \( n^3 - n \) is a product of three consecutive integers, \( n - 1, n, n + 1 \), we have:

\[
\begin{align*}
n^3 - n &= 0 \pmod{6} \\
\Rightarrow F_5 \cdot F_6 \cdot F_7 \cdot F_{13} \text{ divides } F_m
\end{align*}
\]

(by the fact that \( r \) divides \( s \) implies \( F_r \) divides \( F_s \))

Thus, we are done.

Also solved by Paul S. Bruckman, David M. Burton, Frank H. Concliffe, Piero Filipponi, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, and the proposer.

Integral Arithmetic Means

B-601 Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Let \( A_{n,k} = (F_n + F_{n+1} + \cdots + F_{n+k-1})/k \). Find the smallest \( k \) in \{2, 3, 4, \ldots\} such that \( A_{n,k} \) is an integer for every \( n \) in \{0, 1, 2, \ldots\}.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall show that 24 is the value of \( k \) that is being sought.

Our solution will use the following known information:

\[
\begin{align*}
(1) & \quad F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1, \; n \geq 1, \text{ and} \\
(2) & \quad F_{n+t} - F_{n-t} = L_t F_t, \; t \text{ even}.
\end{align*}
\]

[(1) is (I_1) on p. 52 of Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr., Houghton Mifflin, Boston, 1969, and (2) is (I_{24}) on p. 59, ibid.]

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Since
\[ F_n + F_{n+1} + \cdots + F_{n+k-1} = (F_1 + F_2 + \cdots + F_{n+k-1}) - (F_1 + F_2 + \cdots + F_{n-1}) = (F_{n+k+1} - 1) - (F_{n+1} - 1) \] [by (1)]
\[ = F_{n+k+1} - F_{n+1}, \]
\[ A_{n,k} = (F_{n+k+1} - F_{n+1})/k. \]

Let \( n \) be an arbitrary nonnegative integer. If \( k = 24 \),
\[ F_{n+24} - F_{n+1} = F_{(n+1)+12} - F_{(n+1)-12} = L_{n+13}F_{12} \] [by (2)]
\[ = L_{n+13} \cdot 144 \equiv 0 \pmod{24}. \]

Thus, \( A_{n,24} \) is an integer for each nonnegative integer \( n \).

\[ A_{0,2} = (F_3 - F_1)/2 = (2 - 1)/2 = 1/2. \] Proceeding in this same manner, it can be shown that \( A_{0,k} \) is NOT an integer for \( k = 2, 3, 5, 7, 8, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, \) and 23 and that \( A_{1,k} \) is NOT an integer for \( k = 4, 6, 9, 11, \) and 19. Therefore, 24 is the smallest \( k \) in \( \{2, 3, 4, \ldots \} \) such that \( A_{n,k} \) is an integer for every nonnegative integer \( n \).

Also solved by David M. Burton, C. Georghiou, L. Kuipers, H.-J. Seiffert, Sahib Singh, Lawrence Somer, David Zeitlin, and the proposer.

Fibonacci Infinite Series

**B-602** Proposed by Paul S. Bruckman, Fair Oaks, CA

Let \( H_n \) represent either \( F_n \) or \( L_n \).

(a) Find a simplified expression for \( \frac{1}{H_n} - \frac{1}{H_{n+1}} - \frac{1}{H_{n+2}}. \)

(b) Use the result of (a) to prove that
\[ \sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + 2 \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}}. \]

Solution by C. Georghiou, University of Patras, Greece

(a) After some simple algebra it is easy to see that
\[ \frac{1}{H_n} - \frac{1}{H_{n+1}} - \frac{1}{H_{n+2}} = \frac{H_{n+1} - H_nH_{n+2}}{H_nH_{n+1}H_{n+2}} = \frac{H^2_{n+1} - H_nH_{n+2}}{H_nH_{n+1}H_{n+2}}. \]

(b) For \( H_n = F_n \), we have \( F^2_{n+1} - F_n F_{n+2} = (-1)^n \) and since \( F_n = O(n^n) \) it follows that
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} = \sum_{n=1}^{\infty} \left( \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}} - \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}} \right) = \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n+1}F_{2n+2}}. \]

On the other hand, we have
\[ \sum_{n=1}^{\infty} \left( \frac{1}{F_n} - \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}} \right) = -\sum_{n=1}^{\infty} \frac{1}{F_{n}} + \frac{2}{F_{1}} + \frac{1}{F_{2}}. \]
By equating the two sums we get the given expression.

Also solved by Piero Filipponi, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Tzermias, and the proposer.

**Lucas Analogue**

**B-603** Proposed by Paul S. Bruckman, Fair Oaks, CA

Do the Lucas analogue of B-602(b).

Solution by C. Georgiou, University of Patras, Greece

For \( H_n = L_n \), we have \( L_{n+1}^2 - L_n L_{n+2} = 5(-1)^{n+1} \), and since \( L_n = 0(a^n) \) it follows that

\[
\sum_{n=1}^{\infty} \frac{5(-1)^{n+1}}{L_n L_{n+1} L_{n+2}} = 10 \sum_{n=1}^{\infty} \frac{1}{L_{2n-1} L_{2n+1} L_{2n+2}}
\]

On the other hand, we have

\[
\sum_{n=1}^{\infty} \left( \frac{1}{L_n} - \frac{1}{L_{n+1}} - \frac{1}{L_{n+2}} \right) = - \sum_{n=1}^{\infty} \frac{1}{L_n} + \frac{2}{L_1} + \frac{1}{L_2}
\]

By equating the two sums, we get

\[
\sum_{n=1}^{\infty} \frac{1}{L_n} = \frac{7}{3} - 10 \sum_{n=1}^{\infty} \frac{1}{L_{2n-1} L_{2n+1} L_{2n+2}}.
\]

Also solved by Piero Filipponi, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Paul Tzermias, and the proposer.