A NEW EXTREMAL PROPERTY OF THE FIBONACCI RATIO

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0. INTRODUCTION

In some problems in the geometry of numbers and in the theory of diophanting approximation, sequences of lattices play an important role. Especially, it sometimes is very useful to consider the sequence of lattices $(\Gamma_N(\alpha))$, $N \in \mathbb{N}$, where α is a real number and $\Gamma_N(\alpha)$ is the two-dimensional lattice spanned by the vectors $\binom{1/N}{\alpha}$ and $\binom{0}{1}$. See, for example, [2], [9], [10].

It is easy to see that, if α is irrational, then the set of points of $\Gamma_N(\alpha)$ in $\mathbb{R}^2$ will become more and more dense in $\mathbb{R}^2$. We will explain this more exactly and define

$$d(\Gamma) := \sup_{x \in \mathbb{R}^2} \inf_{y \in \Gamma} d(x, y),$$

where $d(x, y)$ denotes the euclidean metric, the "dispersion" of the lattice Γ. (We do this in analogy to the notion of the dispersion of a point-sequence in a metric space; see [4], [5].) Since, by Kronecker's theorem, the sequence $k\alpha$ is dense modulo one, if and only if α is irrational, it is easy to see that

$$\lim_{N \to \infty} d(\Gamma\ (\alpha)) = 0$$

if and only if α is irrational. An obvious question is, what can be said about the speed of convergence of $d(\Gamma_N(\alpha))$ for given α. (Similar questions regarding the dispersion of a sequence have been considered, e.g., in [1], [3], [6], and [7].)

It can be shown that the speed of convergence never can be faster than $O(1/\sqrt{N})$, and that $d(\Gamma_N(\alpha)) = O(1/\sqrt{N})$ if and only if α has bounded continued fraction coefficients. This follows directly from obvious connections of our dispersion d with the dispersion of the sequence $(k/N, \{k\alpha\})$, $k = 1, 2, \ldots, N$, in the unit square and from results on this dispersion in [1] and [3], for example. Thus, it is obvious to ask for which α the value

$$D(\alpha) := \limsup_{N \to \infty} \sqrt{N} \cdot d(\Gamma_N(\alpha))$$

is minimal.

We will find that this problem is quite interesting because it provides a new sort of something like a Markov-spectrum (compare especially with [5]) and a new extremal property of the Fibonacci ratio $\xi = (1 + \sqrt{5})/2$.

Theorem: $\inf D(\alpha) = 1/\sqrt{2}$

$D(\alpha) = 1/\sqrt{2}$ if and only if α is equivalent to $\frac{1+\sqrt{5}}{2}$

$D(\alpha) \geqslant \sqrt{\frac{3\sqrt{3}-1}{8}} = \frac{1.024\ldots}{\sqrt{2}}$ if α is not equivalent to $\frac{1+\sqrt{5}}{2}$.

(Here, "equivalent" is used in the sense of the theory of continued fractions. See Perron [8].)

1. NOTATIONS

The irrational number α is represented by the infinite continued fraction $\alpha := [a_0; a_1, a_2, \ldots]$ and has best approximation denominators $1 = q_0 \leqslant q_1 \leqslant q_2 \leqslant \cdots$ with $q_{\ell+1} = a_{\ell+1}q_\ell + q_{\ell-1}$;

$$\frac{p_i}{q_i} + \frac{\phi_i}{q_i \cdot (q_i + q_{i+1})} = \alpha = \frac{p_i}{q_i} + \frac{\theta_i}{q_i \cdot q_{i+1}} \text{ with } |\theta_i| \leqslant 1 \text{ and } |\phi_i| \geqslant 1.$$

(see [8].) Further we denote

$$s_i := \frac{q_i}{q_{i-1}}; \quad \alpha_i := [a_i; a_{i+1}, a_{i+2}, \ldots]$$

and, for a given fixed N, the index $\ell := \ell(N)$, such that $q^2_{\ell(N)} \leqslant N \leqslant q^2_{\ell(N)+1}$.

We denote the distance of x to the nearest integer by $\|x\|$. For given N and for $r \in \mathbb{N}$, we define

$$M(r) := \left(\left(\frac{r}{N}\right)^2 + \|r\alpha\|^2\right)^{1/2},$$

and again, for given N, we denote by λ_1 and λ_2 the successive minima of Γ_N with respect to the euclidean norm, and also two linearly independent vectors in Γ_N with length λ_1 and λ_2.

F is the parallelogram built by λ_1 and λ_2, and μ is the shorter diagonal, and also its length.

$\xi = (1 + \sqrt{5})/2$ always is the positive Fibonacci ratio.

$\alpha \cong \beta$ means that α is equivalent to β.

2. GENERAL RESULTS

Lemma 1: $d(\Gamma_N) = \frac{1}{2} \cdot \lambda_1 \cdot \lambda_2 \cdot \mu \cdot N$

Proof: Every $x \in \mathbb{R}^2$ lies in one fundamental parallelogram F_x of Γ. Let

$$d_x := \min_{y \in \Gamma} d(x, y),$$

then, by using the triangle inequality, it is easy to see that d_x will be attained for a vertex y of F_x. In the triangle built by the vectors λ_1 and λ_2 and by μ, the angle between λ_1 and λ_2 is between $\pi/3$ and $\pi/2$.

The two other angles in the triangle are less than or equal to this angle. Therefore, the center of the circle through the vertices of this triangle is in the interior or on the boundary of the triangle and $d(\Gamma)$ is equal to the radius of the circle. Thus,

$$d(\Gamma_N) = r = \frac{\lambda_1 \cdot \lambda_2 \cdot \mu}{2 \cdot |F|} = \frac{\lambda_1 \cdot \lambda_2 \cdot \mu \cdot N}{2}. \blacksquare$$

Remark 1: Because of the approximation properties of the q_i, we obviously have for given N: $\lambda_1 = \min_{k \in \mathbb{Z}} M(k) = \min_i M(q_i)$.

Remark 2: For $i \neq j$, the two vectors $[(q_i/N), q_i\alpha - p_i]$ and $[(q_j/N), q_j\alpha - p_j]$ are always linearly independent.

Lemma 2: If $\lambda_1 = M(q_j)$, then $\lambda_2 = M(k)$ with $k = q_{j+1} - c \cdot q_j$ and $0 \leqslant c \leqslant a_{j+1}$.

Proof: If $q_m \leqslant k \leqslant q_{m+1}$ with $m \neq j$, then $M(q_m) \leqslant M(k)$ and, therefore, $k = q_m$. Further, we have

$$\frac{1}{N} = |\det(\lambda_1, \lambda_2)| = \frac{1}{N} \cdot |p_j q_m - p_m q_j|;$$

thus, $m = j + 1$ or $m = j - 1$ (see [8], p. 14).

If $q_{j-1} < k < q_{j+1}$ and if t is the largest intermediate convergent's denominator less than or equal to k, or if t is q_{j-1} if $q_{i-1} \leqslant k < q_{i-1} + q_i$ (i.e., if $t = q_{j+1} - c \cdot q_j$ with a c with $0 < c \leqslant a_{j+1}$), then $M(t) \leqslant M(k)$ and, therefore, $k = t$. $\blacksquare$

Lemma 3: If $\limsup_{i \to \infty} a_i \geqslant 4$, then $D(\alpha) > \dfrac{1.025\ldots}{\sqrt{2}}$.

Proof: In the following, we will write d_N instead of $d(\Gamma_N)$. Then we have

$$\sqrt{N} \cdot d_N = \frac{N^{3/2}\lambda_1\lambda_2\mu}{2} \geqslant \frac{1}{2\sqrt{N}\lambda_1} \quad \text{because } \lambda_1 \cdot \lambda_2 \text{ and } \lambda_1 \cdot \mu \text{ and } \geqslant \frac{1}{N}.$$

Further,

$$N\lambda_1^2 = N \cdot \min_i M(q\ )^2 \leqslant 2 \cdot \min_i \max\left(\frac{q_i^2}{N}, \frac{N}{q_{i+1}^2}\right) = 2 \cdot \max\left(\frac{q_\ell^2}{N}, \frac{N}{q_{\ell+1}^2}\right).$$

If we choose $N = q_\ell \cdot q_{\ell+1}$, then $N \cdot \lambda_1^2 \leqslant 2(q_\ell/q_{\ell+1})$, and so

$$D(\alpha) \geqslant \limsup_{\ell \to \infty} \frac{1}{2\sqrt{2}} \cdot \sqrt{\frac{q_{\ell+1}}{q_\ell}} \geqslant \frac{1}{2\sqrt{2}} \cdot \sqrt{[4;\ 4,\ 1,\ 4,\ 1,\ \ldots]} = \frac{\sqrt{7} + \sqrt{2}}{4}$$

$$= \frac{1.025\ldots}{\sqrt{2}}. \blacksquare$$

Lemma 4: If $\limsup_{i\to\infty} a_i \leqslant 3$, then, for every $k \geqslant 2$ and all ℓ large enough, we have:

a) $M(q_{\ell-k}) > M(q_\ell)$;

b) $M(q_{\ell+k}) > M(q_\ell)$.

Proof of a): It is sufficient to show that $NM^2(q_{\ell-k}) > NM^2(q_\ell)$ with $N = q_\ell^2$, since $NM^2(q_{\ell-k}) - NM^2(q_\ell)$ is monotonically increasing in N for $q_\ell^2 \leqslant N \leqslant q_{\ell+1}^2$.

We always have $q_\ell^2 \cdot \|q_\ell\alpha\|^2 \leqslant 1$, and so

$$q_\ell^2 \cdot M^2(q_{\ell-k}) \geqslant q_\ell^2 \cdot \|q_{\ell-k}\alpha\|^2 \geqslant \left(\frac{q_\ell}{q_{\ell-k} + q_{\ell-k+1}}\right)^2 = \left(\frac{a_\ell q_{\ell-1} + q_{\ell-2}}{q_{\ell-k+1} + q_{\ell-k}}\right)^2$$
$$\geqslant \left(\frac{3}{2}\right)^2 > 2 \geqslant 1 + q_\ell^2 \cdot \|q_\ell\alpha\|^2 = q_\ell^2 \cdot M^2(q_\ell)$$

if $a_\ell \geqslant 2$ or if $k > 2$.

If $a_\ell = 1$ and $k = 2$, then we have to show that

$$\frac{q_{\ell-2}^2}{q_\ell^2} + \frac{q_\ell^2}{q_{\ell-2}\left(s_{\ell-1} + \dfrac{1}{\alpha_\ell}\right)^2} > 1 + \frac{1}{\left(s_{\ell+1} + \dfrac{1}{\alpha_{\ell+2}}\right)^2},$$

because $\|q_\ell\alpha\| = \dfrac{1}{q_\ell\left(s_{\ell+1} + \dfrac{1}{\alpha_{\ell+2}}\right)}$ (see [8], p. 36).

If we write $A := \alpha_{\ell+1}$ and $s := s_{\ell-1}$, then

$$\frac{q_\ell}{q_{\ell-2}} = s + 1,\quad s_{\ell+1} + \frac{1}{\alpha_{\ell+2}} = \frac{s}{s+1} + A,\quad \frac{1}{\alpha_\ell} = \frac{A}{A+1},$$

and

$$\frac{1}{2} + \sqrt{\frac{7}{12}} - \varepsilon_\ell = [1;\, 3,\, 1,\, 3,\, \ldots] - \varepsilon_\ell \leqslant s,\ A \leqslant [3;\, 1,\, 3,\, 1,\, \ldots] + \varepsilon_\ell$$
$$= \frac{3}{2}\left(\sqrt{\frac{7}{3}} + 1\right) + \varepsilon_\ell,$$

with an ε_ℓ with $\lim_{\ell\to\infty} \varepsilon_\ell = 0$.

So it remains to show that, for all A and s in the above region and all ℓ large enough, we have

$$\frac{1}{(s+1)^2} + \frac{(s+1)^2 \cdot (A+1)^2}{(sA+s+A)^2} > 1 + \frac{(s+1)^2}{(s+sA+A)^2},$$

and with $r := s + 1$ and $b := A + 1$, this is equivalent to

$$b^2 - 2b\left(\frac{1}{r} - r\right) - \left(r^2 + 1 - \frac{1}{r^2}\right) > 0,$$

which is true for all

$$b > \frac{1}{r} - r + \sqrt{2r^2 - 1} =: f(r).$$

f is monotonically increasing for $r \geqslant 1/\sqrt{2}$; therefore,

$$f(r) \leqslant f\left(1 + \frac{3}{2} \cdot \left(\sqrt{\frac{7}{3}} + 1\right) + \varepsilon_\ell\right) < \frac{3}{2} + \sqrt{\frac{7}{12}} - \varepsilon_\ell \leqslant b$$

for ℓ large enough, and thus the inequality holds. ∎

Proof of b): Analogous to a), it is sufficient to show that, with $N = q^2_{\ell+1}$, we have $NM^2(q_{\ell+k}) > NM^2(q_\ell)$. We can write

$$q^2_{\ell+1} \cdot \|q_\ell \alpha\|^2 \leqslant 1 \quad \text{and} \quad \frac{1}{s_{\ell+1}} = s_{\ell+2} - a_{\ell+2};$$

therefore,

$$q^2_{\ell+1} \cdot M^2(q_{\ell+k}) > \frac{q^2_{\ell+k}}{q^2_{\ell+1}} \geqslant s^2_{\ell+2} \geqslant (s_{\ell+2} - a_{\ell+2})^2 + 1 = \frac{q^2_\ell}{q^2_{\ell+1}} + 1$$

$$\geqslant q^2_{\ell+1} \cdot M^2(q_\ell). \ ∎$$

3. THE CASE $\alpha \cong \xi$

Lemma 5: If $\alpha \cong \xi$, then, for every ℓ large enough, we have

$$M(q_{\ell+2}) > M(q_{\ell-1}).$$

Proof: It is sufficient to show $NM^2(q_{\ell+2}) > NM^2(q_{\ell-1})$ with $N = q^2_{\ell+1}$. In all that follows, $\varepsilon_\ell(i)$ are reals with $\lim\limits_{\ell\to\infty} \varepsilon_\ell(i) = 0$.

For every $\alpha \cong \xi$, we have

$$\frac{q_{\ell+1}}{q_\ell} = \xi + \varepsilon_\ell(1) \quad \text{and} \quad q_\ell \cdot \|q_\ell \alpha\| = 1/\sqrt{5} + \varepsilon_\ell(2).$$

So, $q^2_{\ell+1} \cdot M^2(q_{\ell+2}) \geqslant \xi^2 + 1/5\xi^2 - \varepsilon_\ell(3) > 1/\xi^4 + \xi^4/5 + \varepsilon_\ell(3) \geqslant q^2_{\ell+1} \cdot M^2(q_{\ell-1})$ for ℓ large enough. ∎

Remark: By Lemmas 2 and 4, λ_1 and λ_2 (not necessarily in this order) will be attained by $M(q_\ell)$ and $M(q_{\ell-1})$ or by $M(q_\ell)$ and $M(q_{\ell+1})$.

In the first case, then, we have for $\alpha \cong \xi$ and for ℓ large enough (because $q_\ell + q_{\ell-1} = q_{\ell+1}$ and $q_\ell - q_{\ell-1} = q_{\ell-2}$):

$$d_N = \frac{N}{2} \cdot \min(M(q_{\ell-1}) \cdot M(q_\ell) \cdot M(q_{\ell+1}),\ M(q_{\ell-2}) \cdot M(q_{\ell-1}) \cdot M(q_\ell)).$$

In the second case, by Lemma 5:

$$d_N = \frac{N}{2} \cdot (M(q_{\ell-1}) \cdot M(q_\ell) \cdot M(q_{\ell+1})).$$

Further, $M(q_{\ell-1}) \cdot M(q_\ell) \cdot M(q_{\ell+1}) \leqslant M(q_{\ell-2}) \cdot M(q_{\ell-1}) \cdot M(q_\ell)$ and, therefore, in any case:

$$d_N = \frac{N}{2} \cdot \min(M(q_{\ell-1}) \cdot M(q_\ell) \cdot M(q_{\ell+1}),\ M(q_{\ell-2}) \cdot M(q_{\ell-1}) \cdot M(q_\ell)).$$

Lemma 6: If $\alpha \cong \xi$, then $D(\alpha) = 1/\sqrt{2}$.

Proof: We write M_k for $M(q_k)$ and set

$$g(N) := N^3 \cdot \min(M_{\ell-2}^2 \cdot M_{\ell-1}^2 \cdot M_\ell^2,\ M_{\ell-1}^2 \cdot M_\ell^2 \cdot M_{\ell+1}^2) = 4Nd_N^2.$$

We have $\lim\limits_{\ell\to\infty}\left|\max\limits_{q_\ell^2 \leqslant N \leqslant q_{\ell+1}^2} g(N) - \max\limits_{1 \leqslant \sigma \leqslant \left(\frac{q_{\ell+1}}{q_\ell}\right)^2} g(\sigma \cdot q_\ell^2)\right| = 0$ and, therefore,

$$4 \cdot (D(\alpha))^2 = \limsup_{\ell\to\infty} \max_{1 \leqslant \sigma \leqslant \left(\frac{q_{\ell+1}}{q_\ell}\right)^2} g(\sigma \cdot q_\ell^2).$$

Now

$$\sigma \cdot q_\ell^2 \cdot M_{\ell-2}^2 \cdot M_{\ell-1}^2 \cdot M_\ell^2 = \left(\frac{q_{\ell-2}^2}{q_\ell^2 \cdot} + \frac{q_\ell^2 \cdot}{5q_{\ell-2}^2}\right) \cdot \left(\frac{q_{\ell-1}^2}{q_\ell^2 \cdot} + \frac{q_\ell^2 \cdot}{5q_{\ell-1}^2}\right) \cdot \left(\frac{q_\ell^2}{q_\ell^2 \cdot} + \frac{q_\ell^2 \cdot}{5q_\ell^2}\right)$$

$$= \left(\frac{1}{\xi^4 \sigma} + \frac{\xi^4\sigma}{5}\right) \cdot \left(\frac{1}{\xi^2\sigma} + \frac{\xi^2\sigma}{5}\right) \cdot \left(\frac{1}{\sigma} + \frac{\sigma}{5}\right) + \varepsilon_\ell(4)$$

$$= \left(\frac{1}{\xi^2\sigma} + \frac{\xi^2\sigma}{5}\right) \cdot \left(\frac{1}{\xi^4\sigma^2} + \frac{\xi^4\sigma^2}{25} + \frac{1}{5\xi^4} + \frac{\xi^2}{5}\right) + \varepsilon_\ell(4)$$

$$= x^3 + x + \varepsilon_\ell(4),$$

with $x = x(\sigma) = 1/\xi^2\sigma + \xi^2\sigma/5$, and quite analogously we get:

$$\sigma \cdot q_\ell^2 \cdot M_{\ell-1}^2 \cdot M_\ell^2 \cdot M_{\ell+1}^2 = y^3 + y + \varepsilon_\ell(5) \text{ with } y = y(\sigma) = \frac{1}{\sigma} + \frac{\sigma}{5}.$$

Consequently, we have (with $\xi_\ell := q_{\ell+1}/q_\ell$):

$$4D^2(\alpha) = \limsup_{\ell\to\infty} \max_{1 \leqslant \sigma \leqslant \xi_\ell^2} \min(x^3 + x + \varepsilon_\ell(4),\ y^3 + y + \varepsilon_\ell(5))$$

$$= \max_{1 \leqslant \sigma \leqslant \xi^2} \min(x^3 + x,\ y^3 + y) = z^3 + z,$$

with $z = \max\limits_{1 \leqslant \sigma \leqslant \xi^2} \min(x(\sigma), y(\sigma))$.

We have $x(\sigma) \geqslant y(\sigma)$ if and only if $\sigma \geqslant \sqrt{5}/\xi$ and, therefore,

$$z = \max\left(\max_{1 \leqslant \sigma \leqslant \sqrt{5}/\xi} x(\sigma),\ \max_{\sqrt{5}/\xi \leqslant \sigma \leqslant \xi^2} y(\sigma)\right) = x(\sqrt{5}/\xi) = 1,$$

and so $D(\alpha) = 1/\sqrt{2}$. ■

4. THE CASE $\alpha \not\sim \xi$

Lemma 7: If a) $m = q_{\ell+2} - c \cdot q_{\ell+1}$ with $0 < c < a_{\ell+2}$
or b) $m = q_\ell - c \cdot q_{\ell-1}$ with $0 < c < a_\ell$,
then $M(m) > M(q_\ell)$.

Proof of a): It is sufficient to show $NM^2(m) > NM^2(q_\ell)$ for $N = q_{\ell+1}^2$.

$$q_{\ell+1}^2 M^2(q_\ell) \geqslant \frac{(q_{\ell-2} - cq_{\ell+1})^2}{q_{\ell+1}^2} = \left(a_{\ell+2} + \frac{q_\ell}{q_{\ell+1}} - c\right)^2 \geqslant \left(\frac{q_\ell}{q_{\ell+1}} + 1\right)^2$$

(continued)

$$> \left(\frac{q_\ell}{q_{\ell+1}}\right)^2 + 1 \geqslant \frac{q_\ell^2}{q_{\ell+1}^2} + q_{\ell+1}^2 \cdot \|q_\ell \alpha\|^2 = q_{\ell+1}^2 \cdot M^2(q_\ell). \blacksquare$$

Proof of b): It is sufficient to show the assertion for $N = q_\ell^2$. We have:

$$\|m\alpha\| = \frac{c}{a_\ell} \cdot \|q_{\ell-2}\alpha\| + \frac{a_\ell - c}{a_\ell} \cdot \|q_\ell \alpha\|.$$

We set $a_\ell = a$, $s_{\ell-1} = s$, $\alpha_{\ell+1} = A$, and we get:

$$q_\ell^2 \cdot M^2(m) > q_\ell^2 \cdot \|m\alpha\|^2 = \left(\frac{c}{a} \cdot \frac{as + 1}{s + \dfrac{A}{aA + 1}} + \frac{a - c}{a} \cdot \frac{1}{A + \dfrac{s}{as + 1}}\right)^2$$

$$> c^2 \geqslant 4 > q_\ell^2 \cdot M^2(q_\ell) \quad \text{if} \quad c \geqslant 2.$$

If $c = 1$, then

$$q_\ell^2 \cdot \|m\alpha\|^2 = \left(\frac{1}{a} \cdot \frac{(as + 1) \cdot (Aa + 1)}{(Aas + a + s)} + \frac{a - 1}{a} \cdot \frac{(as + 1)}{(Aas + a + s)}\right)^2,$$

$$q_\ell^2 \cdot M^2(q_\ell) = 1 + \frac{(as + 1)^2}{(Aas + a + s)^2},$$

and the inequality $q_\ell^2 \cdot \|m\alpha\|^2 \geqslant q_\ell^2 \cdot M^2(q_\ell)$ is, therefore, equivalent to

$$(2Aa^2 - 2Aa - 1) \cdot s^2 + (4Aa - 2A) \cdot s + 2A \geqslant 0.$$

Since $1 = c < a_\ell$, we have $a_\ell \geqslant 2$ and, therefore, because of $A \geqslant 1$, the last inequality holds, and the result is proved. $\blacksquare$

Remark: From all this we have that λ_1 and λ_2 will be attained (not necessarily in this order) by q_ℓ and $q_{\ell-1} + cq$ with $0 \leqslant c \leqslant a_{\ell+1}$.

Lemma 8: $\inf\limits_{\alpha \not\approx \xi} D(\alpha) \geqslant \sqrt{\dfrac{3\sqrt{3} - 1}{8}} = \dfrac{1.024\ldots}{\sqrt{2}}$.

Proof: $\alpha \not\approx \xi$ iff $\limsup\limits_{i \to \infty} a_i > 1$. For $\limsup\limits_{i \to \infty} a_i \geqslant 4$, the result follows from Lemma 3.

First, let $\limsup\limits_{i \to \infty} a_i = 2$ and $a := a_{i+1} = 2$. λ_1 and λ_2 will be attained by q_ℓ and $q_{\ell-1} + cq_\ell$ with $c = 0$, 1, or 2. Therefore, we have

$$4Nd_N^2 \geqslant \min(T_1, T_2, T_3, T_4) =: g(N)$$

with

$$T_1 = N^3 M_{\ell-1}^2 M^2(q_\ell - q_{\ell-1}) M_\ell^2, \quad T_2 = N^3 M_{\ell-1}^2 M_\ell^2 M^2(q_\ell + q_{\ell-1}),$$
$$T_3 = N^3 M_\ell^2 M^2(q_\ell + q_{\ell-1}) M_{\ell+1}^2, \quad T_4 = N^3 M_\ell^2 M_{\ell+1}^2 M^2(q_\ell + q_{\ell+1}).$$

We write $x := q_{\ell-1}/q_\ell$, $A = \alpha_{\ell+1}$, $N = \sigma q_\ell^2$, then again we have

$$\frac{1}{\sqrt{3} + 1} - \varepsilon_\ell(6) \leqslant x \leqslant \sqrt{3} - 1 + \varepsilon_\ell(6)$$

and

$$2 + \frac{1}{\sqrt{3} + 1} - \varepsilon_\ell(7) \leqslant A \leqslant \sqrt{3} + 1 + \varepsilon_\ell(7) \text{ with } \lim_{\ell \to \infty} \varepsilon_\ell(6),\ \varepsilon_\ell(7) = 0$$

because, for example,

$$[0;\ 2,\ 1,\ 2,\ 1,\ \ldots] - \varepsilon_\ell(6) \leqslant x \leqslant [0;\ 1,\ 2,\ 1,\ 2,\ \ldots] + \varepsilon_\ell(6).$$

Further,

$$1 \leqslant \sigma \leqslant \left(\frac{q_{\ell+1}}{q_\ell}\right)^2 = (2 + x)^2,$$

and, for every σ,

$$4D^2(\alpha) \geqslant \limsup_{\substack{\ell \to \infty \\ a_{\ell+1} = 2}} g(\sigma q^2).$$

For every ℓ, we choose $\sigma^2 := \dfrac{(1 + x)(x + A)^2}{(A - 1)}$; thus, we have

$$1 < 2.7\ldots \leqslant \sigma \leqslant 3.47\ldots < (2 + x)^2$$

and

$$T_1 = \left(\frac{x^2}{\sigma} + \frac{\sigma A^2}{(A + x)^2}\right)\left(\frac{(1 + x)^2}{\sigma} + \frac{\sigma(A + 1)^2}{(A + x)^2}\right)\left(\frac{1}{\sigma} + \frac{\sigma}{(x + A)^2}\right)$$

$$= \left(\left(\frac{x}{\sigma} - \frac{\sigma A}{(x + A)^2}\right)^2 + 1\right) \cdot \left(\frac{(1 - x)^2}{\sigma} + \frac{\sigma(A + 1)^2}{(A + x)^2}\right)$$

$$= \frac{1}{x + A}\left((1 - x)^2 \cdot \sqrt{\frac{A - 1}{x + 1}} + (A + 1)^2 \cdot \sqrt{\frac{x + 1}{A - 1}}\right) \cdot \left(\frac{1}{(1 + x)(A - 1)} + 1\right)$$

$$\geqslant \frac{1}{x + A} \cdot \left((1 + x)^2 \cdot \sqrt{\frac{A - 1}{x + 1}} + (A - 1)^2 \cdot \sqrt{\frac{x + 1}{A - 1}}\right) \cdot \left(\frac{1}{(1 + x)(A - 1)} + 1\right)$$

$$= \frac{1}{\sqrt{(1 + x)(A - 1)}} + \sqrt{(1 + x)(A - 1)}$$

$$\geqslant \frac{1}{-\dfrac{1}{\sqrt{3} + 1} + 1} + -\frac{1}{\sqrt{3} + 1} + 1 - \varepsilon_\ell(8) = \frac{3\sqrt{3} - 1}{2} - \varepsilon_\ell(8).$$

And, quite analogously, we get $T_2,\ T_3,\ T_4 \geqslant \dfrac{3\sqrt{3} - 1}{2} - \varepsilon_\ell(9)$; therefore,

$$D(\alpha) \geqslant \frac{3\sqrt{3} - 1}{8}.$$

If $\limsup_{i \to \infty} a_j = 3$ and $a_{\ell+1} = 3$, then $4Nd_N^2 \geqslant \min(T_1,\ T_2,\ T_4,\ T_5,\ T_6)$ with

$$T_5 = N^3 M_\ell^2 M^2 (q_\ell + q_{\ell-1}) M^2 (2q_\ell + q_{\ell-1}) = T_3$$

and

$$T_6 = N^3 M_\ell^2 M^2 (2q_\ell + q_{\ell-1}) M_{\ell+1}^2 = T_2.$$

Now:

$$\sqrt{\frac{7}{12}} - \frac{1}{2} - \varepsilon_\ell(12) \leqslant x \leqslant \frac{1}{\frac{1}{2} + \sqrt{\frac{7}{12}}} + \varepsilon_\ell(10)$$

and

$$\frac{5}{2} + \sqrt{\frac{7}{12}} - \varepsilon_\ell(11) \leqslant A \leqslant 3 + \frac{1}{\frac{1}{2} + \sqrt{\frac{7}{12}}} + \varepsilon_\ell(11).$$

Therefore, in this case, $D(\alpha) \geqslant \frac{1.068\ldots}{\sqrt{2}}$, and the Lemma is proved. ∎

Finally, the Theorem follows from Lemma 6 and Lemma 8.

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