GENERALIZED FIBONACCI CONTINUED FRACTIONS

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(Submitted August 1986)

1. INTRODUCTION

Eisenstein [3] proposed and Lord [8] solved elegant problems to the effect that the infinite continued fractions (in the preferred notation of Khovanskii [7])

$$L_n - \frac{(-1)^n}{L_n} - \frac{(-1)^n}{L_n} - \cdots = \alpha^n, \tag{1.1}$$

where L_n is the n^{th} lucas number and α is the positive root of $x^2 - x - 1 = 0$.

The purpose of this note is to generalize (1.1), which we do in (4.2) for the sequence $\{w_n\} \equiv \{w_n(a, b; p, q)\}$ (see Horadam [5]). This is defined by the initial conditions $w_0 = a$, $w_1 = b$, and the recurrence relation

$$w_n = pw_{n-1} - qw_{n-2}, \; n \geqslant 2, \tag{1.2}$$

where p and q are arbitrary integers.

2. NOTATION

Following Horadam, we let $\alpha = (p + \sqrt{(p^2 - 4q)})/2$, $\beta = (p - \sqrt{(p^2 - 4q)})/2$, with $|\beta| < 1$, be the roots of

$$x^2 - px + q = 0, \tag{2.1}$$

so that $\{w_n\}$ has the general term

$$w_n = A\alpha^n + B\beta^n, \tag{2.2}$$

where $A = (b - a\beta)/d$, $B = (a\alpha - b)/d$, and $AB = e/d^2$ in which $e = pab - qa^2 - b^2$, $d = \alpha - \beta$, $p = \alpha + \beta$, and $q = \alpha\beta$. Furthermore, for notational convenience, let

$$Q_n = ABq^n. \tag{2.3}$$

For example, for the sequence of Fibonacci numbers $\{F_n\} \equiv \{w_n(0, 1; 1, -1)\}$, $Q_n = (-1)^{n+1}/5$; for the Lucas numbers $\{L_n\} \equiv \{w_n(2, 1; 1, -1)\}$, $Q_n = (-1)^n$; and for the Pell numbers $\{P_n\} \equiv \{w_n(0, 1; 2, -1)\}$, $Q_n = (-1)^n/8$.

3. THE CONVERGENTS

Let $x_k = p_k/q_k$ be the k^{th} convergent of the continued fraction (CF)

$$\mathrm{CF}(w_n) = w_n - \frac{Q_n}{w_n -}\,\frac{Q_n}{w_n -}\cdots. \tag{3.1}$$

$$x_k - x_{k+1} = \frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} = (p_k^2 - p_{k+1}p_{k-1})/q_kq_{k+1},$$

since $p_k = q_{k+1}$ (Khinchin [6]). So, from equations (1.9) and (4.3) of [5],

$$x_k - x_{k+1} = Q_n^k/q_kq_{k+1}. \tag{3.2}$$

For further notational convenience, suppose we write

$$X_k = x_{k+1} \tag{3.3}$$

so that (3.2) has the form

$$X_k - X_{k-1} = -Q_n^k/q_kq_{k-1}. \tag{3.4}$$

Replace k by $k+1$ in (3.4) to get

$$X_{k+1} - X_k = -Q_n^{k+1}/q_{k+1}q_k. \tag{3.5}$$

If we add (3.4) and (3.5), then

$$\begin{aligned} X_{k+1} - X_{k-1} &= -\frac{Q_n^k}{q_k}\left(\frac{Q_n}{q_{k+1}} + \frac{1}{q_{k-1}}\right) = -\frac{Q_n^k}{q_k}\left(\frac{q_{k-1}Q_n + q_{k+1}}{q_{k+1}q_{k-1}}\right) \\ &= -\frac{Q_n^k}{q_k}\left(\frac{q_{k-1}Q_n + w_nq_k - Q_nq_{k-1}}{q_{k+1}q_{k-1}}\right) \quad \text{[from (4.3)]} \\ &= -w_nQ_n^k/q_{k+1}q_{k-1}. \end{aligned}$$

Replace k by $2K$, so that

$$X_{2K+1} - X_{2K-1} = -w_nQ_n^{2K}/q_{2K-1}q_{2K+1}. \tag{3.6}$$

Now, by (3.6),

$$\begin{aligned} X_3 - X_1 &= -w_nQ_n^2/q_1q_3 \\ X_5 - X_3 &= -w_nQ_n^4/q_3q_5 \\ &\vdots \\ X_{2K+1} - X_{2K-1} &= -w_nQ_n^{2K}/q_{2K-1}q_{2K+1}. \end{aligned}$$

On adding, we get

$$X_{2K+1} = w_n\left(1 - \frac{Q_n^2}{q_1q_3} - \frac{Q_n^4}{q_3q_5} - \cdots - \frac{Q_n^2}{q_{2K-1}q_{2K+1}}\right) - \frac{Q_n}{w_n} \tag{3.7}$$

since $X_1 = w_n - Q_n/w_n$.

Similarly, on replacing k by $2K - 1$, we obtain

$$X_{2K} = w_n\left(1 - \frac{Q_n}{q_0q_2} - \frac{Q_n^3}{q_2q_4} - \cdots - \frac{Q_n^{2K-1}}{q_{2K-2}q_{2K}}\right) \tag{3.8}$$

since $X_0 = w_n$.

With our notation adapted to Khovanskii's treatment, he established that when all the coefficients w_n and $-Q_n$ are positive:

(i) the convergents of odd order generate a monotonically increasing sequence with upper bound the even convergent $w_n - Q_n/w_n$, that is, $\lim_{K\to\infty} X_{2K}$ exists and is smaller than each even convergent; and

(ii) the convergents of even order generate a monotonically decreasing sequence with lower bound the odd convergent w_n, that is, $\lim_{K\to\infty} X_{2K-1}$ exists and is greater than each odd convergent.

4. THE CONTINUED FRACTION

In either case, the value of the limit is a root of the equation

$$x = w_n - Q_n/x, \tag{4.1}$$

which can be rewritten as

$$\begin{aligned} 0 &= x^2 - w_nx + Q_n \\ &= x^2 - (A\alpha^n + B\beta^n)x - AB\alpha^n\beta^n = (x - A\alpha^n)(x - B\beta^n). \end{aligned}$$

Since $x_k > w_n - 1$, we have

$$(w_n) = A\alpha^n. \tag{4.2}$$

For example, $\mathrm{CF}(L_n) = \alpha^n$, $\mathrm{CF}(F_n) = \alpha^n/d$, $\mathrm{CF}(P_n) = \alpha_1^n\sqrt{2}/4$, and $\mathrm{CF}(L_1) = \alpha$ (see Vorob'ev [9]), where $\alpha = (1 + \sqrt{5})/2$ and $\alpha_1 = 1 + \sqrt{2}$. This is consistent with $|w_n - A\alpha^n| = B|\beta|^n < B$ if $|\beta| < 1$, or $|F_n - \alpha^n/d| < 1/2$ and $|L_n - \alpha^n| < 1/2$ as in Hoggatt [4].

Since $x_k = w_n - Q_n/x_{k-1}$, we have $\dfrac{p_k}{q_k} = w_n - \dfrac{Q_n}{p_{k-1}/q_{k-1}}$ or

$$p_k = w_np_{k-1} - Q_np_{k-2},\ k \geqslant 2, \tag{4.3}$$

with $p_0 = 1$ and $p_1 = w_n$ since $p_k = q_{k+1}$. Note that (3.2) can also be expressed as $p_{k+1}q_k - p_kq_{k+1} = -Q_n^k$ or, in determinantal form, as

$$\begin{vmatrix} p_k & p_{k+1} \\ q_k & q_{k+1} \end{vmatrix} = Q_n^k. \tag{4.4}$$

5. CONCLUDING COMMENTS

It can be seen from the recurrence relation (4.3) and the initial conditions for p_k, that the numerators of the convergents, $\{p_k\} \equiv \{p_k(1, w_n; w_n, Q_n)\}$, form a generalized Fibonacci sequence. The first few terms can be constructed as follows:

$$\begin{array}{ll} p_0 = 1 & p_4 = w_n^4 - 3Q_n w_n^2 + Q_n^2 \\ p_1 = w_n & p_5 = w_n^5 - 4Q_n w_n^3 + 3Q_n^2 w_n \\ p_2 = w_n^2 - Q_n & p_6 = w_n^6 - 5Q_n w_n^4 + 6Q_n^2 w_n^2 - Q_n^3 \\ p_3 = w_n^3 - 2Q_n w_n & p_7 = w_n^7 - 6Q_n w_n^5 + 10Q_n^2 w_n^3 - 4Q_n^3 w_n \end{array}$$

It can be seen that the values of the numerical coefficients seem to satisfy the partial recurrence relation

$$a_{ij} = a_{i-1,j} - a_{i-2,j-1}, \quad i, j \geqslant 0, \tag{5.1}$$

with boundary conditions given by $a_{i0} = 1$ and $a_{ij} = 0$ if $i < 0$ or $j < 0$ or $j > [i/2]$ (the integer part of $i/2$). (Note: i and j refer to row and column numbers, respectively.)

To establish that the numerical coefficients satisfy (5.1), we first solve (5.1) and then show that the solutions in (5.3) can be used to generate p_k in (5.4). Following Carlitz [2], we set (formally)

$$F(x, y) = \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j, \tag{5.2}$$

and rewrite (5.1) using the boundary conditions on a_{ij}:

$$\begin{aligned} F(x, y) &= x \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{i-1,j} x^{i-1} y^j - x^2 y \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{i-2,j-1} x^{i-2} y^{j-1} \\ &= x \sum_{i=0}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j - x^2 y \sum_{i=0}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j \\ &= x + x \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j - x^2 y - x^2 y \sum_{i=1}^{\infty} \sum_{j=0}^{[i/2]} a_{ij} x^i y^j \\ &= x + xF(x, y) - x^2 y - x^2 y F(x, y) \\ &= (x - x^2 y)/(1 - x + x^2 y) \\ &= x(1 - xy)(1 - x(1 - xy))^{-1} \\ &= \sum_{i=0}^{\infty} x^{i+1}(1 - xy)^{i+1} = \sum_{i=1}^{\infty} x^i (1 - xy)^i \\ &= \sum_{i=1}^{\infty} \sum_{j=0}^{i} (-1)^j \binom{i-j}{j} x^i y^j, \end{aligned}$$

whence, on equating coefficients of xy,

$$a_{ij} = (-1)^j \binom{i-j}{j}. \tag{5.3}$$

So, from equation (2.8) of Barakat [1],

$$p_k = \sum_{j=0}^{[k/2]} (-1)^j \binom{k-j}{j} Q_n^j w^{k-2j}, \tag{5.4}$$

and it can be confirmed by induction on k that (5.4) satisfies the recurrence relation (4.3).

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