ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-428 *Proposed by Larry Taylor, Rego Park, NY*

Let j, m, and n be integers. Let a and b be relatively prime even-odd integers with b not divisible by 5. Let $A_n = aL_n + bF_n$. Then $A_n = A_{n+1} - A_{n-1}$ with initial values $A_1 = b + a$, $A_{-1} = b - a$.

Prove that the following three numbers

$$(2F_{n-j}A_{m-j},\ F_{n+j}A_{m+j},\ 2F_{2j}A_{n+m})$$

are in arithmetic progression.

H-429 *Proposed by John Turner, Hamilton, New Zealand*

Fibonacci enthusiasts know what happens when they add two adjacent numbers of a sequence and put the result next in line.

Have they considered what happens if they put the results *in the middle*?

They will get the following increasing sequence of T-sets (multi-sets):

$T_1 = \{1\}$
given initial sets
$T_2 = \{1, 2\}$

$T_3 = \{1, 3, 2\}$,
$T_4 = \{1, 4, 3, 5, 2\}$,
$T_5 = \{1, 5, 4, 7, 3, 8, 5, 7, 2\}$,
$T_6 = \{1, 6, 5, 9, 4, 11, 7, 10, 3, 11, 8, 13, 5, 12, 7, 9, 2\}$,
etc.

Prove that for $3 \le i \le n$ the multiplicity of i in multi-set T_n is $\frac{1}{2}\phi(i)$, where ϕ is Euler's function.

SOLUTIONS

What's the Point?

H-406 *Proposed by R. A. Melter, Long Island U., Southampton, NY and I. Tomescu, U. of Bucharest, Romania (Vol. 25, no. 1, February 1987)*

Let A_n denote the set of points on the real line with coordinates 1, 2, ..., n. If $F(n)$ denotes the number of pairwise noncongruent subsets of A_n, then prove

$$F(n) = \begin{cases} 2^{n-2} + 2^{n/2} - 1 & \text{for even } n, \\ 2^{n-2} + 3 \cdot 2^{(n-3)/2} - 1 & \text{for odd } n. \end{cases}$$

Solution by the proposers

Let $n = n_1 + \cdots + n_k$ be a decomposition of n into k nonnegative parts. It is well known that the number of such decompositions is equal to

$$\binom{n-1}{k-1}$$

The decompositions $n_1 + \cdots + n_k$ and $n_k + \cdots + n_1$ will be said to be conjugate.

It follows that

$$F(n) = 1 + \sum_{m=1}^{n-1} \alpha(m)$$

where $\alpha(m)$ is the number of pairwise nonconjugate decompositions of m.

Denote by $\alpha(m, k)$ the number of pairwise nonconjugate decompositions of m into k parts:

We shall consider four cases:

(i) $k = 2\ell$ and $m = 2p$.

In this case the number of self-conjugate decompositions of m with k parts is equal to

$$\binom{p-1}{\ell-1}$$

and hence,

$$\binom{p-1}{\ell-1} + \binom{p-2}{\ell-1} + \cdots + \binom{\ell-1}{\ell-1} + \binom{p}{\ell}.$$

Thus $\alpha(m, k) = \binom{m-1}{k-1} - \frac{1}{2}\left[\binom{m-1}{k-1} - \binom{p}{\ell}\right] = \frac{1}{2}\binom{2p}{2\ell} + \frac{1}{2}\binom{p}{\ell}.$

In order to calculate $\alpha(m)$, consider two subcases.

I. Let $m = 2p$. It follows that

$$\alpha(m) = \frac{1}{2}\left[\sum_{\ell \ge 0}\binom{2p-1}{2\ell} + \sum_{\ell \ge 0}\binom{p-1}{\ell}\right] + \frac{1}{2}\left[\sum_{\ell \ge 1}\binom{2p-1}{2\ell-1} + \sum_{\ell \ge 1}\binom{p-1}{\ell-1}\right]$$

$$= \frac{1}{2}\left[\sum_{\ell \ge 0}\binom{2p}{2\ell} + \sum_{\ell \ge 0}\binom{p}{\ell}\right] = \frac{1}{2}(2^{2p-1} + 2^p) = 2^{2p-2} + 2^{p-1}$$

$$= 2^{m-2} + 2^{(m-2)/2}.$$

II. Let $m = 2p + 1$. One can write

$$\alpha(m) = \frac{1}{2}\sum_{\ell \ge 1}\binom{2p}{2\ell-1} + \frac{1}{2}\left[\sum_{\ell \ge 0}\binom{2p}{2\ell} + \sum_{\ell \ge 0}\binom{p}{\ell}\right] = \frac{1}{2}\sum_{\ell \ge 0}\binom{2p}{\ell} + \frac{1}{2}\sum_{\ell \ge 0}\binom{p}{\ell}$$

$$= 2^{m-2} + 2^{(m-3)/2}.$$

$$\alpha(m, k) = \binom{m-1}{k-1} - \frac{1}{2}\left[\binom{m-1}{k-1} - \binom{p-1}{\ell-1}\right]$$

$$= \frac{1}{2}\binom{2p-1}{2\ell-1} + \frac{1}{2}\binom{p-1}{\ell-1}.$$

(ii) $k = 2\ell$, $m = 2p + 1$.

Here there are no self-conjugate decompositions; hence,

$$\alpha(m, k) = \frac{1}{2}\binom{m-1}{k-1}.$$

(iii) $k = 2\ell + 1$, $m = 2p$.

In order to count the number of self-conjugate decompositions, observe that the central position (m_ℓ) of $m_1 + \cdots + m_{2\ell+1}$ must be an even integer.

Thus, the number of self-conjugate decompositions is equal to

$$\binom{p-2}{\ell-1} + \binom{p-3}{\ell-1} + \cdots + \binom{\ell-1}{\ell-1} = \binom{p-1}{\ell}.$$

It follows that, in this case

$$\alpha(m, k) = \binom{m-1}{k-1} - \frac{1}{2}\left[\binom{m-1}{k-1} - \binom{p-1}{\ell}\right]$$

$$= \frac{1}{2}\binom{2p-1}{2\ell} + \frac{1}{2}\binom{p-1}{\ell}.$$

(iv) $k = 2\ell + 1$, $m = 2p + 1$.

It can be seen that the central position of a self-conjugate decomposition must be an odd number.

Finally, for odd n,

$$\begin{aligned} F(n) &= 1 + 2^{-1} + 2^{-1} + 2^0 + 2^0 + 2^1 + 2^0 + 2^2 + 2^1 + \cdots + 2^{n-3} + 2^{(n-3)/2} \\ &= 2 + (2^0 + \cdots + 2^{n-3}) + (2^0 + 2^0 + 2^1 + 2^1 + \cdots \\ &\qquad + 2^{(n-5)/2} + 2^{(n-5)/2}) + 2^{(n-3)/2} \\ &= 2^{n-2} + 3 \cdot 2^{(n-3)/2} - 1. \end{aligned}$$

For even n, one obtains

$$\begin{aligned} F(n) &= 1 + 2^{-1} + 2^{-1} + 2^0 + 2^0 + 2^1 + 2^0 + 2^2 + 2^1 + \cdots \\ &\qquad + 2^{n-4} + 2^{(n-4)/2} + 2^{(n-3)} + 2^{(n-4)/2} \\ &= 2 + (2^0 + \cdots + 2^{n-3}) + (2^0 + 2^0 + \cdots + 2^{(n-4)/2} + 2^{(n-4)/2}) \\ &= 2^{n-2} + 2^{n/2} - 1. \end{aligned}$$

Also solved by Paul Bruckman.

Nice End Product

H-407 *Proposed by Paul S. Bruckman, Lynwood, WA (Vol. 25, no. 1, February 1987)*

Find a closed form for the infinite product:

$$\prod_{n=0}^{\infty} \frac{(5n+2)(5n+3)}{(5n+1)(5n+4)}. \quad (1)$$

Solution by Carl Libis, student, Tempe, AZ

Use Theorem 5, p. 14, in Rainville's *Special Functions*, which says:

"If $\sum_{k=1}^{s} a_k = \sum_{k=1}^{s} b_k$, and if no a_k or b_k is a negative integer,

$$\prod_{n=1}^{\infty} \frac{(n + a_1)(n + a_2) \ \dots \ (n + a_s)}{(n + b_1)(n + b_2) \ \dots \ (n + b_s)} = \frac{\Gamma(1 + b_1)\Gamma(1 + b_2) \ \dots \ \Gamma(1 + b_s)}{\Gamma(1 + a_1)\Gamma(1 + a_2) \ \dots \ \Gamma(1 + a_s)}."$$

Thus,

$$\prod_{n=0}^{\infty} \frac{(5n + 2)(5n + 3)}{(5n + 1)(5n + 4)} = \frac{\frac{2}{5}\cdot\frac{3}{5}}{\frac{1}{5}\cdot\frac{4}{5}} \prod_{n=1}^{\infty} \frac{\left(n + \frac{2}{5}\right)\left(n + \frac{3}{5}\right)}{\left(n + \frac{1}{5}\right)\left(n + \frac{4}{5}\right)}$$

$$= \frac{\frac{2}{5}\cdot\frac{3}{5}}{\frac{1}{5}\cdot\frac{4}{5}} \cdot \frac{\Gamma\left(\frac{6}{5}\right)\Gamma\left(\frac{9}{5}\right)}{\Gamma\left(\frac{7}{5}\right)\Gamma\left(\frac{8}{5}\right)}$$

$$= \frac{\frac{2}{5}\cdot\frac{3}{5}}{\frac{1}{5}\cdot\frac{4}{5}} \cdot \frac{\left(\frac{1}{5}\right)\Gamma\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\left(\frac{2}{5}\right)\Gamma\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)\Gamma\left(\frac{3}{5}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)} = \frac{\sin\left(\frac{2\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)} = 2\cos\left(\frac{\pi}{5}\right).$$

Also solved by D. Antzoulakos, O. Brugia & P. Filipponi, C. Georghiou, W. Janous, B. Prielipp, J. Shallit, and the proposer.

Ghost from the Past

H-125 *Proposed by Stanley Rabinowitz, Far Rockaway, NY (Vol. 5, no. 5, December 1967)*

Define a sequence of positive integers to be *left-normal* if given any string of digits, there exists a member of the given beginning with this string of digits, and define the sequence to be *right-normal* if there exists a member of the sequence ending with the string of digits.

Show that the sequences whose n^{th} terms are given by the following are left-normal but not right-normal.

a. $P(n)$, where $P(x)$ is a polynomial function with integral coefficients.
b. P_n, where P_n is the n^{th} prime.
c. $n!$
d. F_n, where F_n is the n^{th} Fibonacci number.

Comment by Chris Long, student, Rutgers U., New Brunswisk, NJ

Left-normality for all of the above was established by Raymond E. Whitney in this journal, vol. 11, no. 1, p. 77, and vol. 11, no. 2, pp. 186-187; he also established that (b) and (c) are not right-normal. For (d), note that the Fibonacci sequence is defective mod 8, and hence is defective mod 1000; this shows that the Fibonacci sequence is not right-normal. However, the statement that no polynomials with integer coefficients are right-normal is false, as the example $P(n) \equiv n$ demonstrates. Indeed, David Moews, a student at Harvard College, came up with the following characterization of right-normal polynomials with integer coefficients.

Theorem (Moews): If Q is a polynomial with integer coefficients, then Q is right-normal iff

(1) for all m there exists n with $Q(n) \equiv m \pmod{10}$,

(2) for all n, $(Q'(n), 10) = 1$, where $Q'(x)$ is the formal derivative of $Q(x)$.

Proof (Moews): Note that for all $m \geq 1$, Q can be viewed as a function from $\mathbb{Z}/10^m\mathbb{Z}$ into $\mathbb{Z}/10^m\mathbb{Z}$. Q will be right-normal just when this function is surjective for all m; since $\mathbb{Z}/10^m\mathbb{Z}$ is finite, this will be the case just when this function is injective for all m. We induce on m to show that (1) and (2) imply this.

If $m = 1$, this is clear; otherwise, let $m > 1$. Suppose we have x, y with $Q(x) \equiv Q(y) \pmod{10^m}$. Then $Q(x) \equiv Q(y) \pmod{10^{m-1}}$, so by the induction hypothesis, $x \equiv y \pmod{10^{m-1}}$. Let $x = y + k10^{m-1}$. Then, since $m > 1$, $2(m-1) \geq m$, so $Q(x) \equiv Q(y) + k10^{m-1}Q'(y) \pmod{10^m}$, and we must have $k10^{m-1}Q'(y) \equiv 0 \pmod{10^m}$, i.e., $kQ'(y) \equiv 0 \pmod{10}$, which gives $k \equiv 0 \pmod{10}$ since $Q'(y)$ is relatively prime to 10. Hence, $x \equiv y \pmod{10^m}$, which completes the induction.

For the other implication, it is clear that Q cannot be right-normal if (1) fails. If (2) fails, let n have $(Q'(n), 10) = a$, $a > 1$. Then, if $b = 10/a$,

$$Q(n + 10b) \equiv Q(n) + 10bQ'(n) \pmod{100},$$

and a divides $Q'(n)$ so 100 divides $10bQ'(n)$, which means that

$$Q(n + 10b) \equiv Q(n) \pmod{100}.$$

This proves that Q is not injective as a function from $\mathbb{Z}/100\mathbb{Z}$ to $\mathbb{Z}/100\mathbb{Z}$, so Q cannot be right normal. Q.E.D.

Examples that David Moews came up with of $Q(x)$'s which satisfy conditions (1) and (2), and that are therefore right-normal, include

$$Q(x) = ax + b \text{ for } (a, 10) = 1$$

and higher degree polynomials such as

$$Q(x) = 2x^5 + 5x^4 + 5x^2 + 9x.$$

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