ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. Hillman

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-634 *Proposed by P. L. Mana, Albuquerque, NM*

For how many integers n with $1 \le n \le 10^6$ is $2^n \equiv n \pmod 5$?

B-635 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

For all positive integers n, prove that

$$2^{n+1}\left[1 + \sum_{k=1}^{n}(k!k)\right] < (n+2)^{n+1}.$$

B-636 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

Solve the difference equation

$$x_{n+1} = (n+1)x_n + \lambda(n+1)^3[n!(n! - 1)]$$

for x_n in terms of λ, x_0, and n.

B-637 *Proposed by John Turner, U. of Waikato, Hamilton, New Zealand*

Show that

$$\sum_{n=1}^{\infty} \frac{1}{F_n + aF_{n+1}} = 1,$$

where a is the golden mean $(1 + \sqrt{5})/2$.

B-638 *Proposed by Herta T. Freitag, Roanoke, VA*

Find s and t as function of k and n such that

$$\sum_{i=1}^{k} F_{n-4k+4i-2} = F_s F_t.$$

B-639 *Proposed by Herta T. Freitag, Roanoke, VA*

Find s and t as function of k and n such that

$$\sum_{i=1}^{k} L_{n-4k+4i-2} = F_s L_t.$$

SOLUTIONS

No Fibonacci Pythagorean Triples

B-610 *Proposed by L. Kuipers, Serre, Switzerland*

Prove that there are no positive integers r, s, and t such that (F_r, F_s, F_t) is a Pythagorean triple (that is, such that $F_r^2 + F_s^2 = F_t^2$).

Solution by Marjorie Bicknell-Johnson, Santa Clara, CA

V. E. Hoggatt, Jr., proved that no three distinct Fibonacci numbers can be the lengths of the three sides of a triangle. (See page 85 of *Fibonacci and Lucas Numbers*, Houghton Mifflin Mathematics Enrichment Series, Houghton Mifflin, Boston, 1969.) Since a Pythagorean triple gives integral lengths for the sides of a right triangle, his result is more general. Hoggatt's elegant proof follows, where a, b, and c are the sides of the triangle:

> In any triangle, we must have $a + b > c$, $b + c > a$, and $c + a > b$. For any three consecutive Fibonacci numbers, $F_n + F_{n+1} = F_{n+2}$, and so there can be no triangle with sides having measures F_n, F_{n+1}, F_{n+2}. In general, consider Fibonacci numbers, F_r, F_s, F_t, where $F_r \le F_{s-1}$ and $F_{s+1} \le F_t$. Since $F_{s-1} + F_s = F_{s+1}$ and $F_r \le F_{s-1}$, we have $F_r + F_s \le F_{s+1}$, and since $F_{s+1} \le F_t$, we have $F_r + F_s \le F_t$. Therefore, there can be no triangle with sides having measure F_r, F_s, and F_t.

Also solved by Charles Ashbacher, Paul S. Bruckman, Piero Filipponi, C. Georghiou, Sahib Singh, Lawrence Somer, and the proposer.

Each Term a Multiple of 3

B-611 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S(n) = \sum_{k=1}^{n} L_{4k+2}.$$

For which positive integers n is $S(n)$ an integral multiple of 3?

Solution by Bob Prielipp, U. of Wisconsin-Oshkosh

We shall show that $S(n)$ is an integral multkple of 3 for each positive integer n.

The claimed result is an immediate consequence of the following lemma.

Lemma: 3 divides L_{4k+2} for each nonnegative integer k.

Proof: Because $L_2 = 3$, the specified result holds when $k = 0$. Let j be a nonnegative integer. Then

$$\begin{aligned} L_{4(j+1)+2} &= L_{4j+6} = L_{4j+4} + L_{4j+5} \\ &= (L_{4j+2} + L_{4j+3}) + (L_{4j+2} + 2L_{4j+3}) = 2L_{4j+2} + 3L_{4j+3}. \end{aligned}$$

Hence, if 3 divides L_{4j+2}, then 3 divides $L_{4(j+1)+2}$. The required result now follows by mathematical induction.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, L. Kuipers, Chris Long, Br. J. M. Mahon, H.-J. Seiffert, Sahib Singh, Lawrence Somer, H. J. M. Wijers, Gregory Wulczyn, and the proposer.

When the Sum Is a Multiple of 7

B-612 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$T(n) = \sum_{k=1}^{n} F_{4k+2}.$$

For which positive integers n is $T(n)$ an integral multiple of 7?

Solution by Lawrence Somer, Washington, D.C.

By inspection, we observe that the period of $\{F_n\}$ modulo 7 is 16. Now,

$$F_2 = 1 \equiv 1 \pmod 7, \quad F_6 = 8 \equiv 1 \pmod 7,$$
$$F_{10} = 55 \equiv -1 \pmod 7, \quad F_{14} = 377 \equiv -1 \pmod 7.$$

It thus follows that

$$F_{4k+2} \equiv 1 \pmod 7 \text{ if } k \equiv 0 \text{ or } 1 \pmod 4$$

and

$$F_{4k+2} \equiv -1 \pmod 7 \text{ if } k \equiv 2 \text{ or } 3 \pmod 4.$$

Consequently, it follows that $T(n)$ is an integral multiple of 7 for a positive integer n if and only if n is an even integer.

Also solved by Paul S. Bruckman, Piero Filipponi, C. Georghiou, Br. J. M. Mahon, Bob Prielipp, H.-J. Seiffert, Sahib Singh, H. J. M. Wijers, Gregory Wulczyn, and the proposer.

Finding the Constants

B-613 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Show that there exist integers a, b, and c such that

$$F_{n+p}^2 + F_{n-p}^2 = aF_n^2F_p^2 + b(-1)^pF_n^2 + c(-1)^nF_p^2.$$

Solution by C. Georghiou, University of Patras, Greece

We will show that $a = 5$ and $b = c = 2$. Indeed, from the identity

$$5F_n^2 = L_{2n} - 2(-1)^n,$$

we find

$$5F_{n+p}^2 + 5F_{n-p}^2 = L_{2n+2p} + L_{2n-2p} - 4(-1)^{n+p} = L_{2n}L_{2p} - 4(-1)^{n+p}$$

and

$$25F_n^2F_p^2 = L_{2n}L_{2p} - 2(-1)^pL_{2n} - 2(-1)^nL_{2p} + 4(-1)^{n+p}.$$

It follows, therefore, that

$$\begin{aligned} F_{n+p}^2 + F_{n-p}^2 - 5F_n^2F_p^2 &= (2(-1)^pL_{2n} + 2(-1)^nL_{2p} - 8(-1)^{n+p})/5 \\ &= 2(-1)^pF_n^2 + 2(-1)^nF_p^2. \end{aligned}$$

Also solved by Paul S. Bruckman, Herta T. Freitag, L. Kuipers, Br. J. M. Mahon, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

Quadruple Products Mod 8

B-614 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy*

Let $L(n) = L_{n-2}L_{n-1}L_{n+1}L_{n+2}$ and $F(n) = F_{n-2}F_{n-1}F_{n+1}F_{n+2}$. Show that

$$L(n) \equiv F(n) \pmod 8$$

and express $[L(n) - F(n)]/8$ as a polynomial in F_n.

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

Using I_{20} and I_{29} in Hoggatt's *Fibonacci and Lucas Numbers*, we get:

$$L(n) = L_n^4 - 25 \quad \text{and} \quad F(n) = F_n^4 - 1.$$

Replacing L_n^2 by $5F_n^2 + 4(-1)^n$, we get

$$L(n) - F(n) = 24F_n^4 + 40(-1)^nF_n^2 - 8 \equiv 0 \pmod 8.$$

Hence,

$$\frac{L(n) - F(n)}{8} = 3F_n^4 + 5(-1)^nF_n^2 - 1.$$

Also solved by Paul S. Bruckman, Herta T. Freitag, C. Georghiou, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Gregory Wulczyn, David Zeitlin, and the proposer.

Identity for Iterated Lucas Numbers

B-615 *Proposed by Michael Eisenstein, San Antonio, TX*

Let $C(n) = L_n$ and $a_n = C(C(n))$. For $n = 0, 1, \ldots,$ prove that

$$a_{n+3} = a_{n+2}a_{n+1} \pm a_n.$$

Solution by C. Georghiou, University of Patras, Greece

It is easy to see that $a_n = \alpha^{L(n)} + \beta^{L(n)}$. Therefore,

$$\begin{aligned} a_{n+2}a_{n+1} &= (\alpha^{L(n+2)} + \beta^{L(n+2)})(\alpha^{L(n+1)} + \beta^{L(n+1)}) \\ &= \alpha^{L(n+3)} + \beta^{L(n+3)} + (-1)^{L(n+1)}(\alpha^{L(n)} + \beta^{L(n)}) \end{aligned}$$

from which the assertion follows.

Also solved by Paul S. Bruckman, Piero Filipponi, Herta T. Freitag, L. Kuipers, Br. J. M. Mahon, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, David Zeitlin, and the proposer.

*****