REPRESENTATIONS FOR REAL NUMBERS VIA k^{th} POWERS OF INTEGERS

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Introduction

The three related classical series for representing real numbers as the sums of reciprocals of integers were all studied toward the end of the nineteenth century. These are, respectively, the series of Sylvester, Engel, and Lüroth (see Perron [1]). More precisely, *given any real number A there exist three (different) sequences of integers $\{a_i\}$ such that*

$$\text{(i)}\quad A = a_1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

where $a_1 \geq 2$, $a_{i+1} \geq a_i(a_i - 1) + 1$ for $i \geq 1$,

$$\text{(ii)}\quad A = a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots,$$

where $a_1 \geq 2$, $a_{i+1} \geq a_i$ for $i \geq 1$,

$$\text{(iii)}\quad A = a_0 + \frac{1}{a_1} + \frac{1}{(a_1 - 1)a_1} \cdot \frac{1}{a_2} + \frac{1}{(a_1 - 1)a_1(a_2 - 1)a_2} \cdot \frac{1}{a_3} + \cdots,$$

where $a_i \geq 2$ for $i \geq 1$.

Observe that as we move from the Sylvester series (i) to the Lüroth series (iii), the denominators in the expansion become increasingly more complex while at the same time the growth conditions on the digits a_i become simpler. We now generalize the expansions in (i) and (ii) above, to obtain new representations for real numbers that depend on a power $k > 0$. These new representations have the desirable property of having terms only slightly more complex than in (i) and (ii) above, yet their digits need satisfy only mild growth conditions. Two different sets of algorithms leading to results of the types mentioned are considered. We state the main results in the case where the digits a_i grow least.

Given any fixed real $k \geq 1$ and any real number A, there exist sequences of integers $\{a_i\}$ such that

$$\text{(i)}\quad A = a_0 + \frac{1}{a_1^k} + \frac{1}{a_2^k} + \frac{1}{a_3^k} + \cdots,$$

where $a_{i+1} \geq a_i \geq 2$ for $i \geq 1$, and for i sufficiently large,

$$a_i + 1 \leq a_{i+1} \leq 2^{1/k} a_i + 1,$$

$$\text{(ii)}\quad A = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1 a_2)^k} + \frac{1}{(a_1 a_2 a_3)^k} + \cdots,$$

where $a_1 = 2$, $1 \leq a_i \leq 2$ for $i \geq 2$, and $a_i = 2$ infinitely often,

(iii) $$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \frac{1}{(a_1 a_2)^k a_3} + \cdots,$$

where $a_1 = 2$, $1 \le a_i \le 2$ for $i \ge 2$, and $a_i = 2$ infinitely often.

Since only the digits 1 and 2 are used, the representations (ii) and (iii) above could be regarded as being analogous in some fashion to the binary representation for real numbers. Expansions of the above form where the digits have no upper bounds are also considered. We note in particular that, by setting $k = 1$ in the above results, we obtain expansions for real numbers with the same form as the Sylvester and Engel series but whose digits are considerably smaller. In addition, when k is a positive integer, *rational* numbers have representations of types (ii) and (iii) above for which the digits a_i become *periodic*. This condition is analogous to that of the Lüroth series when A is rational.

The paper is set out as follows. In Section 2, we consider k^{th} power analogues of the Sylvester series. In Section 3, we consider k^{th} power analogues of the Engel series. Finally, in Section 4, we consider k^{th} power expansions that are related to a simplified version of the Lüroth series.

For convenience we introduce the following notational conventions. The lower case letters a_i and a_n denote *integers* throughout the paper. Furthermore, unless otherwise stated, the lower case letter k represents a positive real number.

2. Generalizations of Sylvester Series

We introduce two different algorithms that lead to a k^{th} power generalization of the series of Sylvester. The first coincides with the ordinary Sylvester algorithm for $k = 1$. The second leads to a restricted growth of the digits in all cases, including $k = 1$.

Theorem 2.1: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1^k} + \frac{1}{a_2^k} + \frac{1}{a_3^k} + \cdots,$$

where:

if $k > 1$, then $a_{i+1} \ge a_i \ge 2$ for $i \ge 1$, and for i sufficiently large,

$a_{i+1} \ge a_i + 1$,

if $0 < k \le 1$, then $a_{i+1} \ge a_i(a_i - 1) + 1$ for $i \ge 1$, $a_1 \ge 2$.

Proof: In order to obtain this result, we introduce the following algorithm:

Given any real number A, let $A_1 = A - a_0$, $0 < A_1 \le 1$.

Then we recursively define

$$a_n = \left[\frac{1}{A_n^{1/k}}\right] + 1 \quad \text{for } n \ge 1,\ A_n > 0,$$

where

$$A_{n+1} = A_n - \frac{1}{a_n^k} \quad \text{for } A_n > 0.$$

First, repeated application of the algorithm yields

$$A = a_0 + A_1 = a_0 + \frac{1}{a_1^k} + A_2 = \cdots = a_0 + \frac{1}{a_1^k} + \frac{1}{a_2^k} + \cdots + \frac{1}{a_n^k} + A_{n+1}.$$

Now $a_n = [1/A_n^{1/k}] + 1$ implies that, for $0 < A_n$, $(a_n - 1)^k \le 1/A_n < a_n^k$. Thus,

$$A_n > \frac{1}{a_n^k},$$

and provided $a_n \ge 2$ $(0 < A_n \le 1)$

$$A_n \le \frac{1}{(a_n - 1)^k}.$$

Now $0 < A_1 \le 1$ implies $a_1 \ge 2$ and

$$A_2 = A_1 - \frac{1}{a_1^k} > \frac{1}{a_1^k} - \frac{1}{a_1^k} = 0.$$

Continuing this process inductively we see that $A_n > 0$ for all n. Furthermore, since $\{A_n\}$ is a strictly decreasing sequence of positive values, we deduce that $a_{n+1} \ge a_n \ge 2$ for $n \ge 1$. Therefore,

$$A_{n+1} = A_n - \frac{1}{a_n^k} \le \frac{1}{(a_n - 1)^k} - \frac{1}{a_n^k} = \frac{a_n^k - (a_n - 1)^k}{(a_n - 1)^k a_n^k}.$$

Thus,

$$a_{n+1}^k > \frac{1}{A_{n+1}} \ge \frac{(a_n - 1)^k a_n^k}{a_n^k - (a_n - 1)^k}.$$

In the case $0 < k \le 1$, $a_n^k - (a_n - 1)^k \le 1$; so

$$a_{n+1} \ge (a_n - 1)a_n + 1, \; n \ge 1.$$

In the case $k > 1$, we have

$$a_{n+1} \ge a_n + 1$$

provided

$$\frac{(a_n - 1)^k}{a_n^k - (a_n - 1)^k} \ge 1.$$

This is true if $a_n \ge 2^{1/k}/(2^{1/k} - 1)$. **and** if $A_n \le (2^{1/k} - 1)^k/2$. On the contrary, suppose that

$$2 \le a_n \le \left[\frac{2^{1/k}}{2^{1/k} - 1}\right] = c(k),$$

say. Then

$$A_{n+1} = A_n - \frac{1}{a_n^k} \le A_n - \frac{1}{(c(k))^k}.$$

Now, either $A_{n+1} \le (2^{1/k} - 1)^k/2$ or

$$A_{n+2} = A_{n+1} - \frac{1}{a_{n+1}^k} \le A_n - \frac{2}{(c(k))^k} \quad \text{if } a_{n+1} \le c(k).$$

Thus, at each stage, A_{n+i} is decreasing by at least a fixed constant, so after a finite number of steps we must reach a stage at which $A_j \le (2^{1/k} - 1)^k/2$. The result for $k > 1$ now follows, since

$$A_{j+n} \le \frac{1}{(a_{j+n} - 1)^k} \le \frac{1}{(c(k) + n - 1)^k} \to 0 \text{ as } n \to \infty.$$

For $0 < k < 1$, $a_{n+1} \ge (a_n - 1)a_n + 1 \ge a_n + 1$ as $a_n \ge 2$; hence,

$$A_n \le \frac{1}{(n+1)^k} \to 0 \text{ as } n \to \infty.$$

A slight modification to the algorithm leads to the following results.

Theorem 2.2: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1^k} + \frac{1}{a_2^k} + \frac{1}{a_3^k} + \cdots,$$

where

$$a_n + 1 \ge a_n \ge 2 \text{ for } n \ge 1,$$

and for n sufficiently large,

$$a_n + 1 \le a_{n+1} \le 2^{1/k} a_n + 1.$$

Proof: We use the same algorithm as previously, except that now we let

$$a_n = \left[\left(\frac{2}{A_n}\right)^{1/k}\right] + 1 \text{ for } n \ge 1,\ A_n > 0.$$

As before, $a_{n+1} \ge a_n \ge 2$ for $n \ge 1$, but now

$$\frac{2}{a_n^k} < A_n \le \frac{2}{(a_n - 1)^k} \text{ if } 0 < A_n \le 1.$$

Therefore, for $n \ge 1$,

$$A_{n+1} = A_n - \frac{1}{a_n^k} > \frac{1}{a_n^k}$$

and

$$a_{n+1} = 1 + \left[\left(\frac{2}{A_{n+1}}\right)^{1/k}\right] < 1 + 2^{1/k} a_n.$$

Also,

$$A_{n+1} \le \frac{2}{(a_n - 1)^k} - \frac{1}{a_n^k} = \frac{2a_n^k - (a_n - 1)^k}{a_n^k (a_n - 1)^k}.$$

So

$$a_{n+1} > \frac{2}{A_{n+1}} \ge \frac{2a_n^k (a_n - 1)^k}{2a_n^k - (a_n - 1)^k},$$

and we have $a_{n+1} \ge a_n + 1$, provided that

$$\frac{2(a_n - 1)^k}{2a_n^k - (a_n - 1)^k} \ge 1.$$

This is easily seen to be true if $a_n \ge 3^{1/k}/(3^{1/k} - 2^{1/k})$ and if

$$A_n < \frac{2(3^{1/k} - 2^{1/k})^k}{3}.$$

An argument similar to that used in the previous proof shows that these conditions must hold after at most a finite number of steps. Thereafter, as before, $A_n \to 0$ as $n \to \infty$, and the result follows.

We note in particular that by setting $k = 1$ in Theorem 2.2 we get an analogue of the Sylvester series

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots,$$

where $a_{n+1} \ge a_n \ge 3$ for $n \ge 1$ and for n sufficiently large

$$a_n + 1 \le a_{n+1} \le 2a_n + 1.$$

This is a much milder growth condition than the condition $a_{n+1} \geq a_n(a_n - 1) + 1$ for $n \geq 1$ of Sylvester. However, under these weaker conditions we no longer obtain uniqueness for the expansions. For example, if we let instead

$$a_n = \left[\left(\frac{m}{A_n}\right)^{1/k}\right] + 1.$$

where $m > 1$ is a fixed constant, we obtain a new expansion for A where the digits satisfy very similar growth conditions.

As a particular case of these expansions, we note that, by definition, the Riemann zeta function $\zeta(k)$ for $k > 1$ has expansion

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots .$$

Euler's well-known formula for $\zeta(2m)$, $m = 1, 2, 3, \ldots$, then yields

$$2^{2m-1}B_m \frac{\pi^{2m}}{(2m)!} = 1 + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \frac{1}{4^{2m}} + \cdots,$$

where B_m is a Bernoulli number.

3. A Generalization of the Engel Series

Using algorithms essentially similar to those introduced in Section 2, we obtain k^{th} power analogues of the Engel series.

Theorem 3.1: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1 a_2)^k} + \frac{1}{(a_1 a_2 a_3)^k} + \cdots,$$

where:

if $k > 1$, then $a_1 \geq 2$, $a_i \geq 1$ for $i \geq 1$, and $a_i \geq 2$ infinitely often,

if $0 < k \leq 1$, then $a_{i+1} \geq a_i \geq 2$ for $i \geq 1$.

Proof: We make use of the following algorithm. Given any real number A, let $A_1 = A - a_0$, $0 < A_1 \leq 1$. Then we recursively define

$$a_n = \left[\frac{1}{A_n^{1/k}}\right] + 1 \text{ for } n \geq 1,\ A_n > 0,$$

where

$$A_{n+1} = a_n^k A_n - 1 \text{ for } A_n > 0.$$

First, repeated application of the above algorithm yields

$$A = a_0 + A_1 = a_0 + \frac{1}{a_1^k} + \frac{A_2}{a_1^k} = \cdots$$

$$= a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1 a_2)^k} + \cdots + \frac{1}{(a_1 a_2 \ldots a_n)^k} + \frac{A_{n+1}}{(a_1 a_2 \ldots a_n)^k}.$$

Again, if $A_n > 0$, we have $A_n^k > 1/a_n$, and if also $a_n \geq 2$, then

$$A_n \leq \frac{1}{(a_n - 1)^k}$$

Now $0 < A_1 \le 1$ implies $a_1 \ge 2$ and $A_2 = a_1^k A - 1 > 0$, thus $a_2 \ge 1$. Continuing the process inductively we see that $A_n > 0$ and hence $a_n \ge 1$ for all n. We consider now the case $k > 1$. Suppose $a_n \ge 2$, then

$$A_{n+1} = a_n^k A_n - 1 \le \frac{a_n^k}{(a_n - 1)^k} - 1 = \left(1 + \frac{1}{a_n - 1}\right)^k - 1 \le 2^k - 1,$$

since we have assumed $a_n \ge 2$. Now, if $A_{n+1} \le 1$, then $a_{n+1} \ge 2$. Otherwise $a_{n+1} = 1$ and $A_{n+2} = A_{n+1} - 1$. Continuing this process, we see that after at most $[2^k - 1]$ steps with

$$a_{n+i} = 1,\ A_{n+i+1} = A_{n+i} - 1 = A_{n+1} - i,$$

we must reach a stage at which

$$A_{n+j} \le 1 \quad \text{and} \quad a_{n+j} \ge 2.$$

We deduce that the sequence $\{A_n\}$ is bounded above by $2^k - 1$ for all n. Furthermore, there exists a sequence of integers $n_1 = 0 < n_2 < n_3 < \dots$ such that

$$0 < A_{n_i+1} \le 1,\ a_{n_i+1} \ge 2,$$

and $a_n = 1$ for all other $n > 1$. Then

$$0 < \frac{A_{n_i+1}}{(a_1 a_2 \dots a_{n_i})^k} \le \frac{1}{2^{k(i-1)}},$$

and so $S_{n_i} \to A$ as $i \to \infty$, where

$$S_n = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1 a_2)^k} + \dots + \frac{1}{(a_1 \dots a_n)^k} = A - \frac{A_{n+1}}{(a_1 \dots a_n)^k}.$$

Now let $n_{i-1} \le n < n_i$. Then $S_{n_{i-1}} \le S_n < S_{n_i}$, and $n \to \infty$ iff $i \to \infty$. So $S_n \to A$ as $n \to \infty$, i.e., the series converges. For the case $0 < k \le 1$, if $a_n \ge 2$, then

$$a_{n+1}^k > \frac{1}{A_{n+1}} \ge \frac{(a_n - 1)^k}{a_n^k - (a_n - 1)^k}, \text{ since } A_{n+1} \le \frac{a_n^k}{(a_n - 1)^k} - 1.$$

Now for $k \le 1$, $a_n^k - (a_n - 1)^k \le 1$ and since $a_1 \ge 2$ we deduce that $a_{n+1} \ge a_n \ge 2$ for all $n \ge 1$. Thus,

$$\frac{A_{n+1}}{(a_1 \dots a_n)^k} \le \frac{1}{2^{kn}} \to 0 \text{ as } n \to \infty,$$

and again the series converges.

This result gives the ordinary Engel series for $k = 1$.

It is possible to further restrict the growth of the digits a_i, so that for $i \ge 1$ they need only take on the values $a_i = 1$ and $a_i = 2$, for any $k \ge 1$.

Theorem 3.2: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1 a_2)^k} + \frac{1}{(a_1 a_2 a_3)^k} + \dots,$$

where:

If $k \ge 1$, then $a_1 = 2$, $1 \le a_i \le 2$ for $i \ge 2$, and $a_i = 2$ infinitely often, If $0 < k < 1$, then $a_1 \geq 2$, $1 \leq a_i \leq 1 + [2^{1/k}]$ for $i \geq 1$, and $a_i \geq 2$ infinitely often.

Proof: We use the same algorithm as in Theorem 3.1 except that now we let

$$A_1 = A - a_0,\ 1 < A_1 \leq 2,$$

and

$$a_n = \left[\left(\frac{2}{A_n}\right)^{1/k}\right] + 1 \text{ for } n \geq 1,\ A_n > 0.$$

As in the previous result, $A_n > 0$ and $a_n \geq 1$ for all $n \geq 1$. Also,

$$A_n > \frac{2}{a_n^k}$$

which implies

$$A_{n+1} = a_n^k A_n - 1 > 1$$

and, in the case $k \geq 1$,

$$a_{n+1} = 1 + \left[\left(\frac{2}{A_{n+1}}\right)^{1/k}\right] < 1 + 2^{1/k} \leq 3.$$

Thus, $1 \leq a_n \leq 2$ for $n \geq 2$, and (since $1 < A_1 \leq 2$) $a_1 \geq 2$. Also provided $a_n = 2$ (the case $a_n > 2$ cannot occur for $n \geq 1$, by the preceding inequalities) we get

$$A_n \leq \frac{2}{(a_n - 1)^k} = 2 \quad \text{and} \quad A_{n+1} \leq \frac{2a_n^k}{(a_n - 1)^k} - 1 = 2^{k+1} - 1,$$

since we assumed $a_n = 2$. Now, in the same way as in the previous theorem, after at most $[2^{k+1} - 1]$ steps of $a_{n+i} = 1$, we must reach a stage at which $A_{n+j} \leq 2$ and $a_{n+j} = 2$. Therefore, the sequence $\{A_n\}$ is bounded above by $2^{k+1} - 1$ for all $n \geq 1$. The convergence of the series for $k \geq 1$ is now shown in exactly the same way as in the previous theorem. The proof for $0 < k < 1$ is exactly the same except that $a_1 \geq 2$ and for $n \geq 1$,

$$1 \leq a_n \leq 1 + [2^{1/k}].$$

In particular, by setting $k = 1$ in Theorem 3.2, we get an analogue of the Engel series

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots,$$

where $a_1 = 2$, $1 \leq a_i \leq 2$ for $i \geq 2$ and $a_i = 2$ infinitely often. Compare this to the growth condition $a_{i+1} \geq a_i \geq 2$ of Engel. Again under these weaker conditions the expansion is not unique. For example, in Section 4 we consider a different algorithm which for $k = 1$ gives another series with the same form and conditions on the digits, as the series noted here.

We note as well that, if we had defined $A_1 = A - a_0$ with $0 < A_1 \leq 1$, as we did in Theorem 3.1, the digits obtained would have satisfied the same conditions as above for $i \geq 2$, but would have had $a_1 > 2$ if $0 < A_1 < 1/2^{k-1}$. The representation thus obtained would no longer be entirely in a "binary" form.

The representation of rational numbers when k takes on integer values 1, 2, 3, ... is also of interest. The condition that holds, i.e., that A is rational if and only if the digits in the expansion eventually become periodic, corresponds to the criterion for the representation of rational numbers via the Lüroth series. The result below applies to both the algorithms of Theorem 3.1 and Theorem 3.2.

Proposition 3.3: Let $k = 1, 2, 3, \ldots$. The digits in the k^{th} power expansions of Theorem 3.1 (or Theorem 3.2) become periodic if and only if A is rational.

Proof: Suppose firstly that $A_1 = p/q$ is rational (with $p, q \in \mathbb{N}$). Then, since $k \in \mathbb{N}$, each A_n is also rational, with

$$\begin{aligned} A_n &= a_{n-1}^k A_{n-1} - 1 = a_{n-1}^k (a_{n-2}^k A_{n-2} - 1) - 1 \\ &= \cdots = a^k A_1 + b = \frac{p_n}{q}, \end{aligned}$$

where $a \in \mathbb{N}$, $b \in \mathbb{Z}$. Now, for the first algorithm (Theorem 3.1) we have

$$0 < A_n = \frac{p_n}{q} \le 2^k - 1.$$

Thus, every

$$A_n \in \left\{\frac{1}{q}, \frac{2}{q}, \frac{3}{q}, \ldots, \frac{(2^k - 1)q}{q}\right\},$$

and so there exist $m, n \in \mathbb{N}$ such that $A_n = A_{n+m}$. Then the algorithm applied to A_{n+m} gives the same successive digits as when applied to A_n, i.e., the digits become periodic. The same argument applies in the case of the second algorithm except that now $0 < A_n = p_n/q \le 2^{k+1} - 1$.

Conversely, suppose that eventually $a_n = a_{n+m}$. If we use the notation

$$X_n = a_0 + \frac{1}{a_1^k} + \frac{1}{(a_1 a_2)^k} + \cdots + \frac{1}{(a_1 \ldots a_{n-1})^k}$$

and let $\alpha_r = (a_1 a_2 \ldots a_r)^k$, and $\alpha_* = \alpha_{n+m-1}/\alpha_{n-1}$, we have

$$\begin{aligned} A = X_n &+ \frac{1}{\alpha_{n-1}}\left\{\left(\frac{1}{a_n^k} + \frac{1}{a_n^k a_{n+1}^k} + \cdots + \frac{1}{a_n^k a_{n+1}^k \ldots a_{n+m-1}^k}\right)\right. \\ &+ \left(\frac{1}{\alpha_*^k a_n} + \frac{1}{\alpha_* a_n^k a_{n+1}^k} + \cdots + \frac{1}{\alpha_* a_n^k a_{n+1}^k \ldots a_{n+m-1}^k}\right) \\ &+ \left.\left(\frac{1}{\alpha_*^2 a_n^k} + \frac{1}{\alpha_*^2 a_n^k a_{n+1}^k} + \cdots\right) + \cdots\right\} \\ = X_n &+ \frac{1}{\alpha_{n-1}}\left(\frac{1}{a_n^k} + \frac{1}{a_n^k a_{n+1}^k} + \cdots + \frac{1}{a_n^k a_{n+1}^k \ldots a_{n+m-1}^k}\right)\left(1 + \frac{1}{\alpha_*} + \frac{1}{\alpha^2} + \cdots\right) \\ = &\text{ a rational number.} \end{aligned}$$

Note that for the ordinary Engel series the condition $a_{n+1} \ge a_n$ implies that for some n sufficiently large $a_{n+i} = a_n$ for all $i \ge 1$.

4. k^{th} Power Series Related to the Lüroth Series

We could at this stage investigate expansions for real numbers whose terms take the form of the terms of the Lüroth series raised to a power. However, we consider instead a similar type of algorithm which leads to an expansion of simpler form, yet where the digits satisfy similar conditions. In particular, by setting $k = 2$ in the results below, we obtain a series expansion for real numbers with the appearance of a simplified Lüroth series.

Theorem 4.1: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \frac{1}{(a_1 a_2)^k a_3} + \frac{1}{(a_1 a_2 a_3)^k a_4} + \cdots,$$

where:

if $k > 1$, then $a_1 \geq 2$, $a_i \geq 1$ for $i \geq 1$, and $a_i \geq 2$ infinitely often,

if $0 < k \leq 1$, then $a_{i+1} \geq a_i \geq 2$ for $i \geq 1$.

Proof: We derive this result from the following algorithm. Given any real number A, let $A_1 = A - a$, $0 < A_1 \leq 1$. Then we recursively define

$$a_n = 1 + \left[\frac{1}{A_n}\right] \text{ for } n \geq 1,\ A_n > 0,$$

where

$$A_{n+1} = a_n^k A_n - a_n^{k-1} \text{ for } A_n > 0.$$

Applying this algorithm repeatedly, we obtain

$$A = a_0 + A_1 = a_0 + \frac{1}{a_1} + \frac{A_2}{a_1^k} = \cdots$$

$$= a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \cdots + \frac{1}{(a_1 \ldots a_{n-1})^k a_n} + \frac{A_{n+1}}{(a_1 \ldots a_n)^k}.$$

Now $a_n = 1 + [1/A_n]$ implies that for $A_n > 0$,

$$A_n > \frac{1}{a_n},$$

and provided $a_n \geq 2$

$$A_n \leq \frac{1}{a_n - 1}.$$

Now $0 < A_1 \leq 1$ implies that $a_1 \geq 2$ and $A_2 = a_1^k A_1 - a_1^{k-1} > 0$; thus $a_2 \geq 1$. Continuing this process inductively, we see that $A_n > 0$; hence, $a_n \geq 1$ for all n. Consider the case $k > 1$. Suppose now that $a_n \geq 2$; then

$$A_{n+1} = a_n^k A_n - a_n^{k-1} \leq \frac{a_n^k}{a_n - 1} - a^{k-1} = \frac{a_n^{k-1}}{a_n - 1}.$$

Now if $A_{n+1} \leq 1$, then $a_{n+1} \geq 2$. Otherwise, $a_{n+1} = 1$ and $A_{n+2} = A_{n+1} - 1$. Continuing this process, we see that after at most $[a_n^{k-1}/(a_n - 1)]$ steps with

$$a_{n+i} = 1,\ A_{n+i+1} = A_{n+i} - 1 = A_{n+1} - 1,$$

we must reach a stage at which $A_{n+j} \leq 1$ and $a_{n+j} \geq 2$. Hence, there exists a sequence of integers $n_1 = 0 < n_2 < n_3 < \ldots$ such that

$$0 < A_{n_i+1} \leq 1,\ a_{n_i+1} \geq 2, \text{ and } a_n = 1$$

for all other $n > 1$. Then

$$0 < \frac{A_{n_i+1}}{(a_1 a_2 \ldots a_{n_i})^k} \leq \frac{1}{2^{k(i-1)}},$$

and so $S_{n_i} \to A$ as $i \to \infty$, where

$$S_n = a_0 + \frac{1}{a_1} + \cdots + \frac{1}{(a_1 \ldots a_{n-1})^k a_n} = A - \frac{A_{n+1}}{(a_1 \ldots a_n)^k}.$$

Now let $n_{i-1} \le n < n_i$. Then

$$S_{n_{i-1}} \le S_n < S_{n_i}, \text{ and } n \to \infty \text{ iff } i \to \infty.$$

So $S_n \to A$ as $n \to \infty$, and the series converges. For the case $0 < k \le 1$, if $a_n \ge 2$ then

$$a_{n+1} > \frac{1}{A_{n+1}} \ge \frac{a_n - 1}{a_n^{k-1}}.$$

Since $k \le 1$, $a_n^{k-1} \le 1$, and as $a_1 \ge 2$ we deduce that $a_{n+1} \ge a_n \ge 2$ for all $n \ge 1$. Thus,

$$\frac{A_{n+1}}{(a_1 \ldots a_n)^k} \le \frac{1}{2^{kn}} \to 0 \text{ as } n \to \infty,$$

and again the series converges to A.

We note that by setting $k = 2$ in Theorem 4.1 we obtain the expansion

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1^2 a_2} + \frac{1}{(a_1 a_2)^2 a_3} + \cdots,$$

where $a_1 \ge 2$, $a_i \ge 1$ for $i \ge 1$, and $a_i \ge 2$ infinitely often. In many ways this could be regarded as a simplified version of the Lüroth series. In addition, we shall show shortly that, as in the Lüroth case, A is rational if and only if the digits in the expansion become periodic.

A second algorithm for $k \ge 1$ leads to a "binary" series of this type where the digits a_i are equal to 1 or 2, for $i \ge 1$.

Theorem 4.2: Let $k > 0$. Every real number A has a representation

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \frac{1}{(a_1 a_2)^k a_3} + \cdots,$$

where:

If $k \ge 1$, then $a_1 = 2$, $1 \le a_i \le 2$ for $i \ge 2$, and $a_i = 2$ infinitely often,

if $0 < k < 1$, then $a_1 = 2$, $1 \le a_{i+1} < 1 + 2a_i^{1-k}$ for $i \ge 1$, and $a_i \ge 2$ infinitely often.

Proof: We use the algorithm of Theorem 4.1 except that now we let $A_1 = A - a_0$, $1 < A_1 \le 2$, and

$$a_n = 1 + \left[\frac{2}{A_n}\right] \text{ for } n \ge 1,\ A_n > 0.$$

In the same way as before, we can show $A_n > 0$ and $a_n \ge 1$ for all $n \ge 1$. Also in this case

$$A_n > \frac{2}{a_n}$$

which implies

$$A_{n+1} = a_n^k A_n - a_n^{k-1} > a_n^{k-1}.$$

It follows that for $k \ge 1$

$$a_{n+1} = 1 + \left[\frac{2}{A_{n+1}}\right] < 1 + \frac{2}{a_n^{k-1}} \le 3.$$

Thus, $1 \le a_n \le 2$ for $n \ge 2$ and, as $1 < A_1 \le 2$, $a_1 = 2$. Also, provided $a_n = 2$ (the case $a_n > 2$ cannot occur from the above), we get

$$A_n \le \frac{2}{a_n - 1} = 2,$$

and

$$A_{n+1} \le \frac{2a_n^k}{a_n - 1} - a_n^{k-1} = 3.2^{k-1},$$

since we assumed $a_n = 2$. Now in the same way as in the previous theorem, after at most $[3.2^{k-1}]$ steps of $a_{n+i} = 1$, we reach a stage at which $A_{n+j} \le 2$ and $a_{n+j} = 2$. The convergence of the series for $k \ge 1$ is now shown in exactly the same way as in the previous theorem. However here, unlike that case, the sequence $\{A_n\}$ is bounded above for all n by a fixed constant as well. The proof for $0 < k < 1$ is the same except that we now have, for $n \ge 1$,

$$A_{n+1} \le 3.2^{k-1} < 3, \text{ and } a_{n+1} < 1 + 2a_n^{1-k}.$$

We consider now the expansion of rational numbers via these algorithms when k is a positive integer. We show that as in the previous section A is rational if and only if A has an expansion in which the digits become periodic.

Proposition 4.3: Let $k = 1, 2, 3, \ldots$. The digits in the k^{th} power expansions of Theorem 4.1 (or Theorem 4.2) become periodic if and only if A is rational.

Proof: First suppose that the expansion is periodic, that is, eventually

$$a_n = a_{n+m}.$$

Then with the notation

$$X_n = a_0 + \frac{1}{a_1} + \frac{1}{a_1^k a_2} + \cdots + \frac{1}{(a_1 \ldots a_{n-2})^k a_{n-1}}$$

and $\alpha_r = (a_1 a_2 \ldots a_r)^k$, $\alpha_* = \alpha_{n+m-1}/\alpha_{n-1}$, we have

$$\begin{aligned} A = X_n &+ \frac{1}{\alpha_{n-1}}\left\{\frac{1}{a_n} + \frac{1}{a_n^k a_{n+1}} + \cdots + \frac{1}{(a_n \ldots a_{n+m-2})^k a_{n+m-1}}\right. \\ &+ \frac{1}{\alpha_* a_n} + \frac{1}{\alpha_* a_n^k a_{n+1}} + \cdots + \frac{1}{\alpha_* (a_n \ldots a_{n+m-2})^k a_{n+m-1}} \\ &\left. + \frac{1}{\alpha_*^2 a_n} \quad \frac{1}{\alpha_*^2 a_n^k a_{n+1}} + \cdots\right\} \\ = X_n &+ \frac{1}{\alpha_{n-1}}\left(\frac{1}{a_n} + \frac{1}{a_n^k a_{n+1}} + \cdots \right. \\ &\left. + \frac{1}{(a_n \ldots a_{n+m-2})^k a_{n+m-1}}\right)\left(1 + \frac{1}{\alpha_*} + \frac{1}{\alpha_*^2} + \cdots\right) \\ = {} & \text{a rational.} \end{aligned}$$

Conversely, suppose $A_1 = p/q$ is rational (with $p, q \in \mathbb{N}$). Then, since $k \in \mathbb{N}$, each A_n is also rational, with

$$\begin{aligned} A_n = a_{n-1}^k A_{n-1} - a_{n-1}^{k-1} &= a_{n-1}^{k-1}\left(a_{n-2}^k A_{n-2} - a_{n-2}^{k-1}\right) - a_{n-1}^{k-1} \\ &= \cdots = a^k A_1 + b = p_n/q. \end{aligned}$$

where $b \in \mathbb{Z}$, a, $p_n \in \mathbb{N}$. Now in the case of the second algorithm (Theorem 4.2)

$$0 < A_n = p_n/q \leq 3.2^{k-1}.$$

Thus, every

$$A_n \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{3.2^{k-1}q}{q}\right\}$$

and we deduce that the expansion becomes periodic in the same way as in Proposition 3.3. In the case of the algorithm of Theorem 4.1, we do not have a fixed bound for A_n. However, when A is rational,

$$a_n = 1 + \left[\frac{1}{A_n}\right] = 1 + \left[\frac{q}{p_n}\right] \leq q + 1$$

as $p_n \geq 1$ for $A_n > 0$. Using the fact that any A_n for which $a_n = 1$ is bounded above by an A_m for which $a_m \geq 2$, and that (for $a_m \geq 2$)

$$A_m \leq \frac{a_m^{k-1}}{a_m - 1},$$

it follows that, for all $n \geq 1$,

$$A_n \leq (q + 1)^{k-1}.$$

Thus, every

$$A_n \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{(q+1)^{k-1}q}{q}\right\},$$

and again we can deduce that the expansion must eventually become periodic.

In summary, we have found new classes of representations for real numbers that are related to the classical series of Sylvester, Engel, and Lüroth. In many cases, the expansions require very mild growth conditions on the digits and share with the Lüroth series the property of begin periodic when a number is rational. Unlike the classical series, however, the expansions for real numbers with $k \neq 1$ are not unique, slightly different algorithms yielding series with the same properties, but with different digits for the same real number A.

Reference

1. O. Perron. *Irrationalzahlen.* New York: Chelsea Publ. Co., 1951.

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