ON SOME SECOND-ORDER LINEAR RECURRENCES

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1. Introduction

Many problems lead to constant coefficient linear recurrences, mostly of the second order, for which explicit solutions are readily available. In some cases, however, one is faced with the problem of solving nonconstant coefficient linear recurrences. Second- and higher-order linear recurrences with variable coefficients cannot always be solved in closed form. The methods available to deal with such cases are very limited. On the other hand, the theory of differential equations is richer in special formulas and techniques than the theory of difference equations. The lack of a simple "change of variable rule," that is, a formula analogous to the differential formula

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx},$$

in the calculus of finite differences, precludes most of these techniques to carry over when we attempt to solve a difference equation.

Of course, in such cases, a step-by-step procedure, starting with the initial values, is always possible. And in many cases it may be the best approach, especially if one needs the value of the independent variable not far from its initial points. However, we frequently ask the question whether the solution may be written in closed form.

When a certain class of second-order linear recurrences was studied, we arrived at a theorem not found anywhere in the literature and which is stated, after some preliminaries, in the next section. In Section 3 we give a proof of the theorem, and its consequences are examined. It is found that a whole class of second-order linear recurrences can be solved in closed form. Finally, an example is given where the theorem is applied.

2. Preliminaries and a Theorem

Let $I = \{\ldots, -1, 0, 1, \ldots\}$ be the set of all integers. The domain of the (complex-valued) functions defined in this paper will be subsets of I of the form $I_N = \{N, N+1, N+2, \ldots\}$ where $N \in I$ (usually $N = 0$ or 1). We are going to consider linear recurrences written in operator form as

$$E^2y + aEy + by = 0 \tag{1}$$

where E is the shift operator, i.e., $Ey = y(n+1)$, a, b, and y are functions on I_N and where $b(n) \neq 0$ for $n \in I_N$. We will also use the notation

$$y'' + ay' + by = 0 \tag{2}$$

where $y' \equiv Ey$, $y'' \equiv E^2y$, and so on, in order to stress the analogy between recurrences and differential equations.

First, we examine the constant coefficient second-order linear recurrences

$$E^2y + \mu Ey + \nu y = 0 \tag{3}$$

where Greek letters will always stand for scalar quantities.

In the elementary exposition of the theory [1] we try solutions of the form $y(n) = \lambda^n$ for some as yet undetermined scalar λ, and we are thus led to the notion of the characteristic polynomial associated with the given equation. In general, we are able to find two linearly independent solutions and hence the general solution. The question arises, however, as to why we try solutions of that particular form. In the more advanced exposition of the theory [3], linear recurrences are treated as a special case of first-order linear systems where the trial solutions $y(n) = \lambda^n$ appear naturally when we attempt to calculate A^n where A is the matrix coefficient of the system.

For the moment, we make the observation that when (3) is premultiplied by E we get

$$E^2(Ey) + \mu E(Ey) + \nu(Ey) = 0, \tag{4}$$

i.e., whenever y is a solution, Ey is also a solution of (3) and, furthermore, the assumption for the existence of solutions of the form $y(n) = \lambda^n$ is equivalent to the statement $Ey = \lambda y$ for some λ.

Next, take the less trivial case of the recurrence

$$aE(aEy) + \mu aEy + \nu y = 0 \tag{5}$$

where $a(n) \neq 0$ for $n \in I_N$. We try to solve (5) as a first-order recurrence (of the Riccati-type) in a. The substitution $a = Eu/u$ leads to the constant coefficient linear recurrence

$$E^2(uy) + \mu E(uy) + \nu(uy) = 0, \tag{6}$$

which has solutions of the form $u(n)y(n) = \lambda^n$ or

$$\frac{u(n+1)y(n+1)}{u(n)y(n)} = \lambda,$$

i.e., $aEy = \lambda y$ for some λ. Note also that if (5) is premultiplied by E and then by a we get

$$aE(aE(aEy)) + \mu aE(aEy) + \nu aEy = 0, \tag{7}$$

i.e., whenever y is a solution, aEy is also a solution of (5).

The above discussion suggests the following.

Theorem: Let L and M be two linear (difference) operators and suppose that $LMy = 0$ whenever $Ly = 0$. Then there exists (at least) a solution y of $Ly = 0$ such that $My = \lambda y$ for some λ.

3. Proof of the Theorem

Let $\{y_1, y_2, \ldots, y_m\}$ be a basis for the null space of L. Then My_i is also in the null space, $i = 1, 2, \ldots, m$ and can be written as a linear combination of the basis, i.e.,

$$My_i = \sum_{k=1}^{m} c_{ik} y_k, \quad i = 1, 2, \ldots, m. \tag{8}$$

Form the matrix $C = [c_{ik}]$ associated with the operator M and let μ be an eigenvector of C^T with associated eigenvalue λ, i.e., $C^T\mu = \lambda\mu$. Now, let

$$y = \sum_{i=1}^{m} \mu_i y_i.$$

Then

$$My = M\left(\sum_{i=1}^{m} \mu_i y_i\right) = \sum_{i=1}^{m} \mu_i My_i = \sum_{i=1}^{m} \mu_i \sum_{k=1}^{m} c_{ik} y_k$$

$$= \sum_{k=1}^{m}\left(\sum_{i=1}^{m}\mu_i c_{ik}\right)y_k = \lambda y. \tag{9}$$

Now let

$$L \equiv E^2 + aE + bI, \tag{10}$$

and

$$M \equiv pE + qI \tag{11}$$

where $b(n)p(n)q(n) \neq 0$ for $n \in I_N$ and I is the identity operator. Since $My = \lambda y$ can always be solved in closed form, the following problem arises:

> "Given a second-order linear operator L (10), find a first-order operator M (11) such that $LMy = 0$ whenever $Ly = 0$."

Although it is not always possible to find such an M, we proceed to deal with the problem and find out what can be said about it.

It is easy to see that

$$LM = p''E^3 + (q'' + ap')E^2 + (aq' + bp)E + bqI$$

and (12)

$$ML = pE^3 + (a'p + q)E^2 + (b'p + aq)E + bqI.$$

Then

$$pLM - p''ML = rL, \tag{13}$$

provided that

$$r = qp - qp'' \tag{14}$$

and

$$a'p'' - ap' - q'' + q = 0; \tag{15}$$
$$b'p'' - bp - aq' + aq = 0. \tag{16}$$

Thus, p and q must satisfy the second-order linear system (15) and (16). Note, however, that (15) can be "summed," since it can be written as

$$\Delta(ap') = \Delta(\Delta + 2I)q \tag{17}$$

where $\Delta = E - I$ is the difference operator. When (17) is premultiplied by Δ^{-1} gives

$$ap' = q' + q + c \tag{18}$$

where c is a constant. Elimination now of q from (16) and (18) gives

$$ab''p''' - a'(aa' - b')p'' + a(aa' - b')p' - a'bp = 0, \tag{19}$$

which is a third-order linear recurrence in p. Solving (19) is a more difficult problem than the original one (10). Note, however, that if $aa' = b'$, (19) is only a two-term recurrence, which means that the recurrence

$$y'' + a'y' + aa'y = 0 \tag{20}$$

can be solved in closed form for any a. We can say something more. From (18) we have

$$a = (q' + q + c)/p', \tag{21}$$

and when the above expression is substituted in (16) we obtain

$$b = (q^2 + cq + d)/pp' \tag{22}$$

where d is a constant. We are, thus, led to the conclusion that the second-order linear recurrences of the form

$$pp'y'' + p(q' + q + \mu)y' + (q^2 + \mu q + \nu)y = 0, \tag{23}$$

where μ, ν are scalar quantities and p, q are arbitrary functions, can be solved in closed form. Finally, note that (20) is a special case of (23), and when q is constant and $p(n) = n$ in (23) we have the Euler-type difference equation [2].

As an application of the above discussion consider the recurrence

$$y(n + 2) - 2(n + 1)y(n + 1) + \left(n + \frac{1}{2}\right)^2 y(n) = 0. \tag{24}$$

Then

$$L \equiv E^2 - 2(n + 1)E + \left(n + \frac{1}{2}\right)^2 I.$$

It is easy to see that

$$L(E - nI)y - (E - nI)Ly = 0.$$

Therefore, the theorem applies for (24) and, consequently, there is (at least) one solution of (24) among the solutions of

$$(E - nI)y = \lambda y,$$

which are

$$y(n) = \lambda(\lambda + 1)(\lambda + 2) \ldots (\lambda + n - 1).$$

Substitution of $y(n)$ into (24) gives

$$\lambda^2 - \lambda + \frac{1}{4} = 0 \quad \text{or} \quad \lambda = \frac{1}{2}.$$

Therefore, one solution of (24) is

$$y_1(n) = \frac{1}{2}\left(\frac{1}{2} + 1\right)\left(\frac{1}{2} + 2\right) \ldots \left(\frac{1}{2} + n - 1\right) \quad \text{or} \quad y\ (n) = \Gamma\left(\frac{1}{2} + n\right),$$

where $\Gamma(\cdot)$ is the Gamma function. The other, linearly independent, solution $y_2(n)$ can be found by the method of the reduction of order.

References

1. S. Goldberg. *Difference Equations*. New York: Wiley & Sons, 1958.
2. C. Jordan. *Calculus of Finite Differences*. New York: Chelsea, 1965.
3. K. S. Miller. *Linear Difference Equations*. New York: W. A. Benjamin, 1968.

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