THE MANN-SHANKS PRIMALITY CRITERION IN THE PASCAL-T TRIANGLE T_3

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1. Introduction

In [1] H. B. Mann and D. Shanks gave a novel criterion for primality in terms of the displaced entries in the Pascal triangle (T_2); the simple description is as follows. Consider the left-justified form of the Pascal triangle and displace the entries in each row two places to the right from the previous row (so that the $n + 1$ entries in row n occupy columns $2n$ to $3n$, inclusive); also, circle the entries in row n which are divisible by n. Then the column number k is a prime if and only if all the entries in column k are circled.

A little experimentation suggests that the result is also true for the Pascal-T triangle T_3 (see the portion of T_3 below), and in what follows we show that this is the case. [T_m here is the Pascal-T triangle of order m, as defined in Section 2, and the n^{th}-row, k^{th}-column entry is denoted by $C_m(n, k)$; T_2 is *the* Pascal triangle, and

$$C_2(n, k) = \binom{n}{k}.]$$

That is, the same displacement by two is applied to successive rows of T_3, the entries to be circled are chosen in the same way, and it is still true that the column number k is a prime if and only if all the entries in column k are circled.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1																	
1			1	1	1													
2					1	2	3	2	1									
3							1	3	6	7	6	3	1					
4									1	4	10	16	19	16	10	4	1	
5											1	5	15	30	45	51	45	30
6													1	6	21	50	90	126
7															1	7	28	77
8																	1	8

The Displaced Array for T_3

In addition to the original paper of Mann and Shanks, the result for T_2 is also given in Honsberger [2, p. 3] with a slightly different proof. Gould [3] gives yet another version using a theorem of Hermite [and also extends the result to certain arbitrary rectangular arrays, e.g., Fibonomial coefficients, in which the entries satisfy a relation analogous to

$$\binom{n}{k} = n(n - 1) \cdots (n - k + 1)/k!$$

for the binomial coefficients]. All three proofs are straightforward and essentially depend only on the facts that

$$C_2(n,\ k) = \binom{n}{k}$$

has an explicit formula, and the simple property that

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1},$$

so that if n and k are relatively prime, n divides $\binom{n}{k}$.

These simplifications, however, are not available for $C_m(n,\ k)$ with $m > 2$, and so, at least for the present, we show only that T_3 has the property claimed, since in this case the reduction formula given in Section 2 allows us to use only ordinary binomial coefficients.

2. Preliminaries

To keep the exposition here self-contained, we will briefly recall the definition of the Pascal-T triangle T_m (as used e.g., in Bollinger [4]), and state two theorems which are used in the sequel.

Definition: For any $m \geq 0$, T_m is the array whose rows are indexed by $n = 0, 1, 2, \ldots$, and columns by $k = 0, 1, 2, \ldots$, and whose entries are obtained as follows:

(a) T_0 is the all-zero array;

(b) T_1 is the array all of whose rows consist of a one followed by zeros;

(c) T_m, $m \geq 2$, is the array whose $n = 0$ row is a one followed by zeros, whose $n = 1$ row is m ones followed by zeros, and any of whose entries in subsequent rows is the sum of the m entries just above and to the left in the preceding row.

The entry in row n and column k is denoted by $C_m(n,\ k)$, although we note that

$$C_2(n,\ k) = \binom{n}{k},$$

since T_2 is *the* Pascal triangle. There are $(m - 1)n + 1$ nonzero entries in row n, and these are the coefficients in the expansion

$$(1 + x + x^2 + \cdots + x^{m-1})^n = \sum_{k=0}^{(m-1)n} C_m(n,\ k)x^k.$$

The reduction formula referred to earlier is also from [4, Th. 2.2], and its statement is as follows.

Theorem I: $C_m(n,\ k) = \displaystyle\sum_{j=0}^{n} \binom{n}{j} C_{m-1}(j,\ k - j).$

Lastly, we will also need a theorem of Ricci [5]; since this is of some interest in its own right, and the source may not be widely available, we include the short proof.

Theorem II (Ricci): If $a, b, \ldots, c$ are nonnegative integers, and

$$n = a + b + \cdots + c,$$

then

$$\frac{n!}{a!b!\ \dots\ c!} \equiv 0, \quad \bmod\left(\frac{n}{D(a,\ b,\ \dots,\ c)}\right),$$

where $D(a, b, \dots, c)$ denotes the greatest common divisor of $a, b, \dots, c$.

Proof: We first note that for any $k \geq 1$ it follows from $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ that

$$\binom{n}{k} \equiv 0, \quad \bmod\left(\frac{n}{D(n,\ k)}\right).$$

If we now write the left side of the main congruence as

$$\frac{n!}{a!(n-a)!} \cdot \frac{(n-a)!}{b!\ \dots\ c!}$$

and use the fact just noted, we conclude that the first factor here is divisible by $n/D(n, a)$. Considering a similar decomposition for $b, \dots, c$, we conclude that the multinomial coefficient is divisible by the least common multiple of the numbers

$$\frac{n}{D(n,\ a)},\ \frac{n}{D(n,\ b)},\ \dots,\ \frac{n}{D(n,\ c)}.$$

And then by known divisibility properties this least common multiple is

$$\frac{n}{D(D(n,\ a),\ D(n,\ b),\ \dots,\ D(n,\ c))},$$

which is the modulus used in the statement of the theorem.

3. Proof of the Criterion for T_3

In the displaced array for T_3, constructed as described previously, we can of course dispose of the even (composite) column numbers exceeding 2 by noting that the construction puts an uncircled 1 (the first entry in any row) in every such column in the same manner as that for T_2. When k is odd, then, we need to show that, if k is a prime, every entry in column k is circled, and if k is composite, at least one entry in column k is uncircled.

We should also note at the outset that by the construction for the displaced array, the $2n + 1$ entries in row n will now occur in columns $2n$ to $4n$ ($2n \leq k \leq 4n$, or $k/4 \leq n \leq k/2$), and the general entry in position (n, k) will be $C_3(n, k - 2n)$. Then, if $k = p$, p a prime > 3, the entries in column p are the numbers $C_3(n, p - 2n)$ for $p/4 \leq n \leq p/2$, and for these values of n, n and $p - 2n$ are relatively prime.

We now show that for *any* relatively prime n and k, $C_3(n, k)$ is divisible by n [which means that the entries $C_3(n, p - 2n)$ referred to in the previous paragraph will all be circled]. From Theorem I with $m = 3$, we have that

$$\begin{aligned} C_3(n,\ k) &= \sum_{j=0}^{n}\binom{n}{j}C_2(j,\ k-j) = \sum_{j=0}^{n}\binom{n}{j}\binom{j}{k-j} \\ &= \sum_{j=0}^{n}\binom{n}{j}\binom{j}{2j-k} = \sum_{\begin{cases} j = \frac{k+1}{2}, & k \text{ odd} \\ j = \frac{k}{2}, & k \text{ even}\end{cases}}^{k} \frac{n!}{(n-j)!(k-j)!(2j-k)!}. \end{aligned}$$

But from the facts that the arguments of the denominator factorials must add up to n, and that twice the second argument plus the third must add up to k, it follows that the assumption that the arguments have a common divisor $d > 1$ implies that d also divides n and k, contrary to the hypothesis. Theorem II now implies that every term in the sum is therefore divisible by n, and then so is $C_3(n, k)$. Thus, all the entries $C_3(n, p - 2n)$ in column p will be circled in the displaced array.

Finally, for k odd and composite, we let p be an odd prime divisor of k and let $k = p(2r + 1)$. In this case, the row $n = pr$ contributes the entry $C_3(pr, p)$ to column k, where, again from Theorem I,

$$C_3(pr, p) = \sum_{j=\frac{p+1}{2}}^{p} \frac{(pr)!}{(pr - j)!(p - j)!(2j - p)!}.$$

Here, for each term except the last, the assumption that the arguments of the denominator factorials have a common divisor $d > 1$ leads to the conclusion that the prime p is composite; thus, their gcd is 1, and from Ricci's Theorem we again conclude each of these terms is divisible by pr. The last term ($j = p$), however, is just the binomial $\binom{pr}{p}$, which is not divisible by pr [2, p. 8]. Thus, $C_3(pr, p)$ is not divisible by pr, and so there will be an uncircled entry in column k. This completes the proof.

Theorem: In the displaced array for T_3, the column number is a prime if and only if all entries in the column are circled.

Lastly, we note that, as with T_3, a little experimentation suggests the conjecture that the criterion is true in all triangles T_m, but the nature of various formulas for $C_m(n, k)$ (see [4], [6], and [7]) appears to require an approach different from that used here.

References

1. H. B. Mann & D. Shanks. "A Necessary and Sufficient Condition for Primality, and Its Source." *J. Combinatorial Theory (A) 13* (1972):131-134.
2. Ross Honsberger. *Mathematical Gems II.* Washington, D.C.: Mathematical Association of America, 1976.
3. H. W. Gould. "A New Primality Criterion of Mann and Shanks and Its Relation to a Theorem of Hermite With Extension to Fibonomials." *Fibonacci Quarterly 10.4* (1972):355-364, 372.
4. R. C. Bollinger. "A Note on Pascal-*T* Triangles, Multinomial Coefficients, and Pascal Pyramids." *Fibonacci Quarterly 24.2* (1986):140-144.
5. G. Ricci. "Sui Coefficienti Binomiale E Polinomali." *Giornale di Matematiche (Battaglini) 69* (1931):9-12.
6. J. E. Freund. "Restricted Occupancy Theory—A Generalization of Pascal's Triangle." *Amer. Math. Monthly 63.1* (1956):20-27.
7. R. C. Bollinger. "Fibonacci *k*-Sequences, Pascal-*T* Triangles, and *k*-in-a-Row Problems." *Fibonacci Quarterly 22.2* (1984):146-151.

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