THE CONVOLVED FIBONACCI EQUATION

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In this note we consider the recurrence relation

$$f_{n+1} = \sum_{k=0}^{n} f_k a_{n-k} + b_n, \; n = 0, 1, 2, \ldots, \tag{1}$$

where $f \; = 1$ and $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences of parameters. Equation (1) is termed the convolved Fibonacci equation because of the occurrence on the right side of the convolution of the sequences $\langle f_n \rangle$ and $\langle a_n \rangle$. Special cases of (1) include the following:

When $a_0 = a_1 = 1$, $a_n = 0$ for $n \geq 2$, and $b_n = 0$ for $n = 0, 1, 2, \ldots$, the f_n are the usual Fibonacci numbers.

When $a_0 = a_1 = \cdots = a_{r-1} = 1$, $a_n = 0$ for $n \geq r$, and $b_n = 0$ for $n = 1, 2, \ldots$, the f_n are r^{th}-order Fibonacci numbers (see, e.g., [2] and [3]).

When $a_0 = a_1 = 1$, $a_n = 0$ for $n \geq 2$, $b_0 = 0$, and

$$b_1 = \sum_{j=1}^{k} \alpha_j (n + 1)^j \text{ for } n = 1, 2, 3, \ldots,$$

(1) becomes the recent recurrence studied by Asveld [1].

We first take the generating function of (1) to obtain the generating function

$$F(z) = \sum_{n=0}^{\infty} f_n z^n$$

of $\langle f_n \rangle$ in terms of the generating functions $A(z)$ of $\langle a_n \rangle$ and $B(z)$ of $\langle b_n \rangle$. Using standard results (see, e.g., [4]), we immediately get

$$F(z) = \frac{1 + zB(z)}{1 - zA(z)} \tag{2}$$

for all z for which $F(z)$, $A(z)$, $B(z)$ exist and $1 - zA(z) \neq 0$.

Two examples of (1) and their solution via (2) are now presented. The a_n and b_n are integers in the first example, while they are not in the second.

Let a_n and b_n be the usual Fibonacci numbers. In this case, the f_n are called the convolved Fibonacci numbers. Since

$$A(z) = B(z) = \frac{z}{1 - z - z^2},$$

it follows from (2) that

$$F(z) = 1 + \frac{-\sqrt{2}/2}{1 - (\sqrt{2} - 1)z} + \frac{\sqrt{2}/2}{1 + (\sqrt{2} + 1)z}$$

and hence $f_0 = 1$,

$$f_n = \frac{\sqrt{2}}{2}(\sqrt{2} - 1)^n + \frac{\sqrt{2}}{2}(\sqrt{2} + 1)^n, \; n = 1, 2, 3, \ldots .$$

Example 2: A standard six-sided fair die has three sides painted red, two sides painted black, and one side painted white. A series of throws of the die is made. We will determine the probability f_n that nowhere in the first n throws of the die is a throw of black followed by a throw of white.

Let E_n denote the event that nowhere in the first n throws of the die is a throw of black followed by a throw of white, W_n be the event that a white is thrown on throw n, and R_n that a red is thrown on throw n. $\overline{W}_n$ will denote complementation, i.e., the event that a white is not thrown on throw n. We may thus write

$$P(E_n) = P(E_n | W_n)P(W_n) + P(E_n | \overline{W}_n)P(\overline{W}_n)$$

from which

$$f_n = 5/6 \; f_{n-1} + 1/6 \; P(E_n | W_n), \; n = 2, 3, \ldots, \tag{3}$$

where $f_1 = 1$. But

$$\begin{aligned} P(E_k | W_k) &= P(R_{k-1}; E_{k-2}) + P(W_{k-1}; E_{k-1}) \\ &= 1/2 \; f_{k-2} + P(E_{k-1} | W_{k-1})P(W_{k-1}), \; k = 2, 3, \ldots . \end{aligned}$$

Hence,

$$P(E_k | W_k) = 1/2 \; f_{k-2} + 1/6 \; P(E_{k-1} | W_{k-1}), \; k = 2, 3, \ldots, \tag{4}$$

where $f_0 = 1$ and $P(E_1 | W_1) = 1$. Substitution of (4) into (3) for $k = n$ yields

$$f_n = 5/6 \; f_{n-1} + 1/6[1/2 \; f_{n-2} + 1/6 \; P(E_{n-1} | W_{n-1})], \; n = 2, 3, \ldots, \tag{5}$$

for which $P(E_{n-1} | W_{n-1})$ may be found from (4).

Successive substitution of $P(E_k | W_k)$ into (3) for $k = n - 1, \ldots, 1$ yields

$$f_n = 5/6 \; f_{n-1} + 1/2 \sum_{j=0}^{n-2} (1/6)^{n-1-j} f_j + (1/6)^n, \; n = 2, 3, \ldots . \tag{6}$$

Equation (6) and the initial conditions can be expressed in the form of (1) with

$$a_0 = 5/6,$$

$$a_n = 1/2 \; (1/6)^n, \; n = 1, 2, 3, \ldots,$$

$$b_n = 1/6 \; (1/6)^n, \; n = 0, 1, 2, \ldots .$$

References

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3. H. Gabai. "Generalized Fibonacci k-Sequences." *Fibonacci Quarterly 8.1* (1970):31-38.
4. L. G. Mitten, G. L. Nemhauser, & C. Beightler. "A Short Table of z-Transforms and Generating Functions." *Operations Research* 9 (1961):574-578.

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