GENERATING PARTITIONS USING A MODIFIED GREEDY ALGORITHM

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Let A be an increasing sequence of integers with first element 1. The "greedy" algorithm for partitioning an integer n with respect to A is:

1. Choose the largest $a \in A$ such that $a \le n$.

2. Form $n - \left[\frac{n}{a}\right]a$.

3. Repeat this process until n is reduced to 0.

This produces a partition of n and the process is called the *greedy* algorithm since n is reduced by the largest possible bites. In this paper we will deal with what we call the "modified greedy" algorithm which replaces the first step above with

1*. Choose *any* $a \in A$ such that $a \le n$.

Note that this method allows us the flexibility of choosing which elements to remove from n, but once chosen, they must be removed as many times as possible. Therefore, there were many different partitions of n using this algorithm.

Let p_{mn} represent the number of modified greedy partitions of n with largest member m. Then

$$p_{nn} = p_{n-1,\,n} = p_{1n} = p_{2n} = 1, \quad n > 1.$$

Define

$$p_n^* = \sum_1^n p_{in} \quad \text{and} \quad p_0^* = 1. \tag{1}$$

Theorem: $p_{mn} = p_q^*$, where $q \equiv n \pmod{m}$, and $0 < q < m$.

Proof: Every partition counted in p_{mn} contains copies of m by the modified greedy algorithm. Removal of m's from each partition does not change their number but reduces their size so that $p_{mn} = p_q^*$, where $q \equiv n \pmod{m}$. □

The following two equations are corollaries.

$$p_{m,\,n+an} = p_{mn}, \quad a \ge 0. \tag{2}$$

$$p_{m+a,\,2m-1+a} = p_{m-1}^*, \quad a \ge 0. \tag{3}$$

Table 1 exhibits p_{mn} with $A = \{1, 2, 3, \ldots\}$ and $m, n = 1, 2, 3, \ldots, 15$. Equations (2) and (3) describe patterns evident in the table. Note that the n^{th} row has n positive entries, $p_{mn} = p_q^*$, in which q is a maximum corresponding to $m = (n+1)/2$, $[(n/2) + 1]$ if n is odd [even].

The following conjectures are derived from a larger (80 × 80) table. (Define $\Delta p_n^* = p_{n+1}^* - p_n^*$.)

1. $\log p_n^*$ approximates a linear function of n if $n > 40$.

2. If n is even, $\Delta p_n^* > 0$.

3. If $\Delta p^*_{n-1} < 0$, then n is even and has at least three prime factors.

   Examples: n = 12, 18, 24, 30, 36, 40, 42, 48, 54, 56, 60, 64, 66, 70, 72, 76, and 80.

4. If $\Delta p^*_{n-1} < 0$, then $\Delta p^*_{an-1} < 0$, $a > 0$.

5. For a given n, let m_r, $r = 1, 2, 3, \ldots$ be elements of the set $\{[n/r]\}$, then $m_{r-1} \geq m_r$. Let $q_r \equiv n \pmod{m_r}$ and $m_{r-1} > m_r - j \geq m_r$, in which $j = 0, 1, 2, \ldots, m_{r-1} - m_r - 1$, then $p_{m_r - j, n} = p^*_{q_r \neq (r_j)}$.

   Example: Let $n = 29$, then $m_1 = 29$, $m_2 = 14$, $m_3 = 9$, $m_4 = 7$, ...

   We have $q_2 = 1$; thus, for $j = 0, 1, 2, 3, 4$, we have

$$p_{14-j,29} = p^*_{1+2j}.$$

TABLE 1

Number of Partitions p_{mn}

$n \backslash m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	p^*_n
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	2
3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	3
4	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	4
5	1	1	2	1	1	0	0	0	0	0	0	0	0	0	0	6
6	1	1	1	2	1	1	0	0	0	0	0	0	0	0	0	7
7	1	1	1	3	2	1	1	0	0	0	0	0	0	0	0	10
8	1	1	2	1	3	2	1	1	0	0	0	0	0	0	0	12
9	1	1	1	1	4	3	2	1	1	0	0	0	0	0	0	15
10	1	1	1	2	1	4	3	2	1	1	0	0	0	0	0	17
11	1	1	2	3	1	6	4	3	2	1	1	0	0	0	0	25
12	1	1	1	1	2	1	6	4	3	2	1	1	0	0	0	24
13	1	1	1	1	3	1	7	6	4	3	2	1	1	0	0	32
14	1	1	2	2	4	2	1	7	6	4	3	2	1	1	0	37
15	1	1	1	3	1	3	1	10	7	6	4	3	2	1	1	45

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