ON A CLASS OF DETERMINANTS

Andrew Granville
Queen's University, Kingston, Ontario, Canada, K7L 3N6
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Recently,* D. H. Lehmer posed the following problem:

If \( \sigma_n \) is the coefficient of \( x^n \) in \((1 + x + x^2)^n \), then show that
\( 2^n \) is the determinant of the matrix

\[
M_n = \begin{bmatrix}
\sigma_0 & \cdots & \sigma_n \\
\sigma_1 & \cdots & \sigma_{n+1} \\
\vdots & \ddots & \vdots \\
\sigma_n & \cdots & \sigma_{2n}
\end{bmatrix}.
\]

He noted that the generating function for the \( \sigma_n \)'s is
\[
(1 - 2x - 3x^2)^{-1/2} = 1 + x + 3x^2 + 7x^3 + 19x^4 + \ldots.
\]

One might equally ask the value of the same determinant where the \( \sigma_n \)'s are the coefficients of \( x^n \) in \((a + bx + cx^2)^n \) [note that these \( \sigma_n \)'s have generating function \((1 - 2bx + dx^2)^{-1/2} \), where \( d = b^2 - 4ac \)]; or perhaps where the \( \sigma_n \)'s are the coefficients of \( x^n + r \) in \((a + bx + cx^2)^n \) for some fixed integer \( r \).

As an example, consider the case where the \( \sigma_n \)'s are the coefficients of
\( x^n + r \) in \((1 + 2x + x^2)^n = (1 + x)^{2n} \), that is,

\[
\sigma_n = \binom{2n}{n + r}.
\]

There does not seem to be an immediate combinatorial argument for finding the determinant even in this case.

In this paper we will answer all of these questions in a very simple way, by easy manipulations of the defining polynomials of the \( \sigma_n \)'s. We make the following definitions:

Let \( S \) be the set of sequences of polynomials \( F = [F_n(x)]_{n \geq 0} \) such that each \( F_n(x) \) has degree less than or equal to \( 2n \), and such that \( F_n(x)/x^n \) is symmetric (about \( x^0 \)). [Clearly \( F_n(x) = (1 + x + x^2)^n \) and \( F_n(x) = (1 + x)^{2n} \) are examples of such sequences.] We define the "elementary sequence" of \( S \) to be

\[
I = [I_n(x)]_{n \geq 0},
\]

where \( I_0(x) = 1 \) and \( I_n(x) = x^{2n} + 1 \) for each \( n \geq 1 \).

Suppose \( F, G \in S \) and \( r \) is a fixed integer. For each integer \( n \geq 0 \), let \( A_n(F, G) \) be the \((n + 1)\) by \((n + 1)\) matrix with \((i, j)^{th} \) entry

\[
F_i(x)/x^i \cdot G_j(x)/x^j \quad (\text{for } 0 \leq i, j \leq n).
\]

For any matrix \( A \) with entries in \( \mathbb{Z}[x] \), we define \( c_r(A) \) to be the matrix formed from \( A \) by replacing each entry with the coefficient of \( x^r \). We let \( D_r(A) \) be the determinant of \( c_r(A) \).

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Finally, we let $B_n(F)$ be the $(n + 1)$ by $(n + 1)$ matrix with $(i, j)^{th}$ entry $b_{i,j} (0 \leq i, j \leq n)$, where

$$F_i(x)/x^i = b_{i,0} + \sum_{j=1}^{i} b_{i,j}(x^j + x^{-j}).$$

We will see that the value $D_r[A_n(F, G)]$ is easily computed in terms of the determinants of $B_n(F)$, $B_n(G)$, and $D_r[A_n(I, I)]$.

**Lemma 1:** Suppose that $A$, $U$, and $V$ are $n \times n$ matrices, where $A$ has entries from $\mathbb{C}[x]$ and $U$ and $V$ from $\mathbb{C}$. Then, for any integer $r$,

$$\sigma_0(UAV) = U\sigma_r(A)V.$$

The proof of this lemma follows immediately from the observation that, if $\alpha(x), b(x) \in \mathbb{C}[x]$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha$ times the coefficient of $x^r$ in $\alpha(x)$ plus $\beta$ times the coefficient of $x^r$ in $b(x)$ equals the coefficient of $x^r$ in $\alpha \alpha(x) + \beta b(x)$.

We also make the following trivial observation

**Lemma 2:** If $F, G \in S$, then for any positive integer $n$,

$$A_n(F, G) = B_n(F)A_n(I, I)B_n(G)^T.$$

Combining Lemmas 1 and 2, we observe

**Corollary 1:** If $F, G \in S$ and $r$ is a given integer, then

$$D_r[A_n(F, G)] = D_r[A_n(I, I)] \cdot \text{Det}[B_n(F)] \cdot \text{Det}[B_n(G)].$$

Observing that, by definition, $B_n(F)$ is a lower triangular matrix with diagonal entries $F_m(0)$, $0 \leq m \leq n$, we have

**Lemma 3:** If $F \in S$, then $\text{Det}[B_n(F)] = \prod_{m=0}^{n} F_m(0)$.

We now compute the values of $D_r[A_n(I, I)]$.

**Lemma 4:** For integers $r$ and $n$ with $n \geq 0$, we have

$$D_r[A_n(I, I)] = \begin{cases} 2^n & \text{if } r = 0 \\ (-1)^{(n+1)/2} & \text{if } r \neq 0 \text{ and } 2r \text{ divides } n + 1 \text{ or } n + r, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof:** $\sigma_r[A_n(I, I)]$ has $(i, j)^{th}$ entry equal to the coefficient of $x^r$ in $(x^i + x^{-i})(x^j + x^{-j})$ for $i, j \geq 1$. Thus,

$$\sigma_r[A_n(I, I)] = c_r[A_n(I, I)],$$

so we will assume henceforth that $r \geq 0$. Now, if $r = 0$,

$$[c_0(A_n(I, I))]_{i,j} = \begin{cases} 1 & i = j = 0, \\ 2 & i = j > 0, \\ 0 & \text{otherwise}, \end{cases}$$

and so it is clear that $D_0[A_n(I, I)] = 2^n$.

Let $X = \sigma_r[A_n(I, I)]$ and $D_n = D_r[A_n(I, I)]$. For $r \geq 0$,
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\[
(\lambda)_{i,j} = \begin{cases} 
1 & \text{if } i + j = p, \\
1 & \text{if } |i - j| = r, \\
0 & \text{otherwise.}
\end{cases}
\]

We will prove the result for fixed \( r \) by induction on \( n \).

Now if \( 0 \leq n \leq r - 1 \), then all entries of the top row of \( X \) are zero, and so \( D_n = 0 \). If \( n = r \), then \( X \) has ones on the reverse diagonal and zeros everywhere else, so that

\[
D_n = (-1)^{[(n+1)/2]}
\]

For \( r + 1 \leq n \leq 2r - 2 \), observe that the \( r - 1 \)st and \( r + 1 \)st rows of \( X \) are both \((0, 1, 0, \ldots, 0)\) so that \( D_n = 0 \).

Now let \( K_r \) be the \( 2r \) by \( 2r \) matrix with \( r \times r \) block structure

\[
\begin{bmatrix}
O_r & I_r \\
I_r & O_r
\end{bmatrix}
\]

so that \( \det K_r = (-1)^r \).

If \( n = 2r - 1 \), then the \( i \)th row of \( X \) has all zero entries except for ones in columns \( r - i \) and \( r + i \) if \( i \leq r - 1 \), and in column \( i - r \) if \( i \geq r \). We subtract row \( r + i \) from row \( r - i \) for \( i = 1, 2, \ldots, r - 1 \), which are all determinant-preserving operations and get the matrix \( K_r \). Thus,

\[
D_n = \det K_r = (-1)^{(n+1)/2}.
\]

Now suppose \( n \geq 2r \). If \( i \geq n - r + 1 \), then row \( i \) has just one nonzero entry (in column \( j = i - r \)) and so we can subtract this row from all other rows with entries in the \((i - r)\)th column. (This is clearly a determinant-preserving operation.) We perform the same action for each column \( j \), with \( j \geq n - r + 1 \) and we are left with the matrix

\[
\begin{bmatrix}
Y & 0 \\
0 & K_r
\end{bmatrix}
\]

where \( Y = a_r[A_{n-2r}(I, I)] \).

Thus,

\[
D_n = D_{n-2r}, \quad \det K_r = (-1)^{(n-2r+1)/2}(-1)^r = (-1)^{(n+1)/2}
\]

by the induction hypothesis.

So by combining Corollary 1 with Lemmas 3 and 4, we may state the main

**Theorem:** If \( F, G \in \mathcal{S} \) and \( A \) is the \((n+1)\) by \((n+1)\) matrix whose \((i,j)\)th entry is the coefficient of \( x^{i+j+r} \) in \( F_i(x) \cdot G_j(x) \), then the determinant of \( A \) equals

\[
\prod_{m=0}^{n} F_m(0)G_m(0), \begin{cases} 
2^n & \text{if } r = 0, \\
(-1)^{(n+1)/2} & \text{if } r \neq 0 \text{ and } 2 \text{ divides } n + 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Some consequences are

**Corollary 2:** The determinant of \( M_n \) with \( a_n \) equal to the coefficient of \( x^n \) in \((1 + x + x^2)^n \) is \( 2^n \).

**Proof:** Take \( F_n(x) = G_n(x) = (1 + x + x^2)^n \) in the Theorem.

**Corollary 3:** The determinant of \( M_n \) with \( a_n = \left[\frac{2n}{n + r}\right] \) is

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\[ \begin{cases} 2^n & \text{if } r = 0, \\ (-1)^{(n+1)/2} & \text{if } r \neq 0 \text{ and } 2r \text{ divides } n + 1 \text{ or } n + r, \\ 0 & \text{otherwise.} \end{cases} \]

Proof: Take \( F_m(x) = G_m(x) = (1 + x)^{2m} \) in the Theorem.

We make an interesting combinatorial observation in

Corollary 4: If \( c_n \) is the coefficient of \( x^n \) in \((1 + tx + x^2)^n \), then the value of the determinant of \( M_n \) is independent of \( t \).

Proof: Take \( F_m(x) = G_m(x) = (1 + tx + x^2)^m \) in the Theorem and observe that each \( F_m(0) \) is independent of \( t \).

Corollary 5: The determinant of \( M_n \) with \( c_n \) equal to the coefficient of \( x^{n+r} \) in \((\alpha + bx + \alpha x^2)^n \) (with \( \alpha, b, c \neq 0 \)) is:

\[ (\alpha^{-r}c_{n+r}(n+1)/2)^{2n} \begin{cases} 2^n & \text{if } r = 0, \\ (-1)^{(n+1)/2} & \text{if } r \neq 0 \text{ and } 2^n \text{ divides } n + 1 \text{ or } n + r, \\ 0 & \text{otherwise.} \end{cases} \]

Proof: Let \( \theta = (\alpha c)^{1/2}, x = \theta y/\alpha \), so that \( c_n \) is the coefficient of

\[ \frac{\theta n^{n+r}y^{n+r}}{\alpha^{n+r}} \]

in \( \alpha^n[1 + (b/\theta)y + y^2]^n \). Let \( d_n \) be the coefficient of \( y^{n+r} \) in \([1 + (b/\theta)y + y^2]^n \) so that \( c_n = (\alpha^{-r}c_{n+r})^{1/2} d_n \). Then

\[
\begin{bmatrix}
c_0 & c_1 & \cdots & c_n \\
c_1 & c_2 & \cdots & c_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
c_n & \cdots & c_{2n}
\end{bmatrix} = (\alpha/2)^{r/2} \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \theta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \theta^n
\end{bmatrix} \begin{bmatrix}
d_0 & d_1 & \cdots & d_n \\
d_1 & d_2 & \cdots & d_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
d_n & \cdots & \cdots & d_{2n}
\end{bmatrix} \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \theta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \theta^n
\end{bmatrix},
\]

and so the result follows immediately from Corollaries 3 and 4.

Corollary 6: The Legendre polynomials \([P_n(t)]_{n \geq 0}\) are defined by

\[ (1 - 2tx + x^2)^{-1/2} = \sum_{n \geq 0} P_n(t)x^n. \]

By taking \( c_n = P_n(t) \), the determinant of \( M_n \) is

\[ 2^n \left( \frac{t^2 - 1}{4} \right)^{n(n+1)/2}. \]

Proof: Use Corollary 5 with \( b = t \) and \( b^2 - 4ac = 1 \).

Clearly, this technique of computing this class of determinants may be generalized to a number of different questions. The real keys to the method are that \((1, x + x^{-1}, x^2 + x^{-2}, \ldots)\) form an additive basis for \( \mathbb{Z}[x + x^{-1}] \) over \( \mathbb{Z} \); and that the action of taking the coefficients of \( x^n \) of the entries of a matrix of polynomials, commutes with multiplication by matrices with entries in \( \mathbb{F} \) (i.e., Lemma 1).

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