UNIQUE FIBONACCI FORMULAS

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I. Introduction

In this paper we consider the generating function

$$G(x)^{-k} = 1/(1 - a_1x - a_2x^2 - \cdots - a_mx_m)^k \quad \text{(where } m \geq 2 \text{ and } k \geq 1\text{)}$$

as a formal power series. Note that we can write the expression as

$$G(x)^{-k} = F_{m,k}(0) + F_{m,k}(1)x + F_{m,k}(2)x^2 + \cdots + F_{m,k}(n)x^n + \cdots \tag{1}$$

$$\text{(for } n \geq 0\text{, and where } F_{m,k}(0) = 1\text{)}.$$

However, before considering equation (1), we shall develop certain identities by the use of partitions. Let $p(n)$ denote the number of partitions of n; that is, the number of solutions of the equation

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n$$

in nonnegative integers. We state the following identity established in [1]:

$$p(n) = -\sum_{\substack{0 \leq i < m \\ m < j \leq n}} p(i)e(j - i)p(n - j) \tag{2}$$

$$\text{where } \begin{cases} e(k) = (-1)^k & \text{if } k = (1/2)(3h^2 \pm h)\text{, where } h \text{ is an integer,} \\ e(k) = 0 & \text{otherwise,} \end{cases}$$

and $p(0) = 1$.

The proof of (2) will be evident as a special case of a more general form to be given later. Let

$$g(x) = \sum_{n=0} a(n)x^n \tag{2a}$$

and

$$g(x)^{-1} = \sum_{n=0} b(n)x^n \tag{2b}$$

where, for convenience, $a(0) = b(0) = 1$. Then it can be shown that

$$\sum_{j=0}^{n} a(j)b(n - j) = 0 \quad \text{(for } n > 0\text{)}. \tag{3}$$

For the sums

$$S = \sum_{\substack{0 \le i < m \\ m < j \le n}} a(i)b(j - i)a(n - j) \quad \text{and} \quad T = \sum_{0 \le i \le j \le m} a(i)b(j - i)a(n - j)$$

where $0 < m < n$, using (3) above, it can be shown that

$$T = \sum_{j=0}^{m} a(n - j) \sum_{i=0}^{j} a(i)b(j - i) = a(n). \tag{4}$$

Furthermore,

$$S + T = \sum_{0 \le i \le m} \sum_{i \le j \le n} a(i)b(j - i)a(n - j) = \sum_{0 \le i < m} a(i) \sum_{s=0}^{n-i} b(s)a(n - i - s).$$

Note also that the inner sum on the extreme right vanishes unless $n - 1 = 0$, because $m < n$. Hence, we have $S + T = 0$.

Combining this with (4), we get $S = -a(n)$, or, explicitly,

$$S = \sum_{\substack{0 \le i < m \\ m < j \le n}} a(i)b(j - i)a(n - j) = -a(n) \quad (0 < m < n). \tag{5}$$

Since we may equally well have started out with $g(x)^{-1}$, rather than $g(x)$, we also have

$$S = \sum_{\substack{0 \le i < m \\ m < j \le n}} b(i)a(j - i)b(n - j) = -b(n) \quad (0 < m < n).$$

II. Some Relations Involving Fibonacci and Tribonacci Numbers

Referring to (1), we first examine what happens when $k = 1$ and the $a_j = 1$, for $i \le j \le m$, where $m \ge 2$. For convenience, we let $F_{m,1}(n) = F_m(n)$, where $m \ge 2$ and $n \ge 0$. note that $F_m(0) = 1$, $F_2(n)$ denotes the n^{th} Fibonacci numbers, $F_3(n)$ denotes the n^{th} Tribonacci numbers, etc. Letting $n = 2Z$, for $Z \ge 1$ and $Z > m$, we have

$$F_m(2Z) = F_m(2Z - 1) + F_m(2Z - 2) + F_m(2Z - 3) + \cdots + F_m(2Z - m). \tag{6}$$

Combining (6) with the results of (4) and (5) for the coefficients, we have the following table of values that are used to determine $F_m(2Z)$.

TABLE 1

a	b	1	2	3	...	m
Z	$F_m(Z)$	$F_m(Z-1)$	$F_m(Z-2)$	$F_m(Z-3)$	...	$F_m(Z-m)$
Z+1	$F_m(Z-1)$		$F_m(Z-1)$	$F_m(Z-2)$	...	$F_m(Z-m+1)$
Z+2	$F_m(Z-2)$			$F_m(Z-1)$	...	$F_m(Z-m+2)$
					...	
Z+m-1	$F_m(Z-m+1)$					$F_m(Z-m+(m-1))$

Multiplying the elements in column 2 by the sum of the corresponding elements in each row, we have:

$$\begin{aligned}F_m(2Z) &= F_m(Z)(F_m(Z-1) + F_m(Z-2) + F_m(Z-3) + \cdots + F_m(Z-m))\\ &\quad + F_m(Z-1)(F_m(Z-1) + F_m(Z-2) + \cdots + F_m(Z-m+1))\\ &\quad + F_m(Z-2)(F_m(Z-1) + F_m(Z-2) + \cdots + F_m(Z-m+2)\\ &\quad \vdots\\ &\quad + F_m(Z-m+1)(F_m(Z-m+(m-1))). \qquad (7)\end{aligned}$$

When $m = 2$, we get the Fibonacci numbers, and (7) becomes

$$F_2(2Z) = F_2(Z)(F_2(Z-1) + F_2(Z-2)) + F_2(Z-1)(F_2(Z-1)), \qquad (8)$$

or

$$F_2(2Z) = (F_2(Z))^2 + (F_2(Z-1))^2, \text{ for } Z \geq 1. \qquad (9)$$

When $m = 3$, we get the Tribonacci numbers, and (7) becomes

$$\begin{aligned}F_3(2Z) &= F_3(Z)(F_3(Z-1) + F_3(Z-2) + F_3(Z-3))\\ &\quad + F_3(Z-1)(F_3(Z-1) + F_3(Z-2))\\ &\quad + F_3(Z-2)(F_3(Z-1)), \qquad (10)\end{aligned}$$

so that

$$F_3(2Z) = (F_3(Z))^2 + (F_3(Z-1))^2 + 2F_3(Z-1)F_3(Z-2), \text{ for } Z \geq 1. \qquad (11)$$

Continuing the process of (8)-(11), with $m = 2a$, we have:

$$\begin{aligned}F_{2a}(2Z) &= (F_{2a}(Z))^2 + (F_{2a}(Z-1))^2 + \cdots + (F_{2a}(Z-a))^2\\ &\quad + 2F_{2a}(Z-1)(F_{2a}(Z-2) + F_{2a}(Z-3) + \cdots + F_{2a}(Z-(2a-1)))\\ &\quad + 2F_{2a}(Z-2)(F_{2a}(Z-3) + F_{2a}(Z-4) + \cdots + F_{2a}(Z-(2a-2)))\\ &\quad \vdots\\ &\quad + 2F_{2a}(Z-(a-1))(F_{2a}(Z-a) + F_{2a}(Z-(2a-(a-1)))), \qquad (12)\\ &\qquad a \geq 1 \text{ and } Z \geq 1.\end{aligned}$$

Furthermore,

$$F_{2a}(0) = 1,\ F_{2a}(1) = 1,\ F_{2a}(2) = 2,\ \ldots,\ F_{2a}(2a) = 2^{2a-1}.$$

Continuing the process for $m = 2a + 1$, we have:

$$\begin{aligned}F_{2a+1}(2Z) &= (F_{2a+1}(Z))^2 + (F_{2a+1}(Z-1))^2 + \cdots + (F_{2a+1}(Z-a))^2\\ &\quad + 2F_{2a+1}(Z-1)(F_{2a+1}(Z-2) + F_{2a+1}(Z-3) + \cdots + F_{2a+1}(Z-2a))\\ &\quad + 2F_{2a+1}(Z-2)(F_{2a+1}(Z-3) + F_{2a+1}(Z-4) + \cdots\\ &\qquad + F_{2a+1}(Z-(2a-1)))\\ &\quad \vdots\\ &\quad + 2F_{2a+1}(Z-(a-1))(F_{2a+1}(Z-a) + F_{2a+1}(Z-(a+1))\\ &\qquad + F_{2a+1}(Z-(a+2)))\\ &\quad + 2F_{2a+1}(Z-a)(F_{2a+1}(Z-(a+1))), \qquad (13)\end{aligned}$$

for $a \geq 1$, $Z \geq 1$. Here,

$$F_{2a+1}(0) = 1,\ F_{2a+1}(1) = 1,\ F_{2a+1}(2) = 2,\ F_{2a+1}(3) = 4,\ \ldots,$$

and $F_{2a+1}(2a+1) = 2^{2a}$.

For $n = 2Z + 1$, we also consider $F_{2a}(2Z + 1)$ and $F_{2a+1}(2Z + 1)$, where $Z \geq a$ and $Z \geq 1$. In the exact way we obtained (12) and (13) but with added induction, we now get

$$\begin{aligned}F_{2a}(2Z+1) = {} & (F_{2a}(Z+1))^2 - (F_{2a}(Z-a))^2 - (F_{2a}(Z-(a+1)))^2 - \cdots \\ & - (F_{2a}(Z-(2a-1)))^2 \\ & - [2F_{2a}(Z-(2a-1))(F_{2a}(Z-1) + F_{2a}(Z-2) + F_2\ (Z-3) + \cdots \\ & + F_{2a}(Z-(2a-2)))] \\ & - [2F_{2a}(Z-(2a-2))(F_{2a}(Z-2) + F_{2a}(Z-3) + \cdots \\ & + F_{2a}(Z-(2a-3)))] \\ & \vdots \\ & - [2F_{2a}(Z-(a+2))(F_{2a}(Z-(a-2)) + F_{2a}(Z-(a-1)) \\ & + F_{2a}(Z-a) + F_{2a}(Z-(a+1)))] \\ & - [2F_{2a}(Z-(a+1))(F_{2a}(Z-(a-1)) + F_{2a}(Z-a))] \end{aligned} \tag{14}$$

and

$$\begin{aligned}F_{2a+1}(2Z+1) = {} & (F_{2a+1}(Z+1))^2 - (F_{2a+1}(Z-(a+1)))^2 - \cdots \\ & - (F_{2a+1}(Z-2a))^2 \\ & - [2F_{2a+1}(Z-2a)(F_{2a+1}(z-1) + F_{2a+1}(Z-2) + F_{2a+1}(Z-3) + \\ & \cdots + F_{2a+1}(Z-(a\ -1)))] \\ & - [2F_{2a+1}(Z-(2a-1))(F_{2a+1}(Z-2) + F_{2a+1}(Z-3) + \cdots \\ & + F_{2a+1}(Z-(2a-2)))] \\ & \vdots \\ & - [2F_{2a+1}(Z-(a+2))(F_{2a+1}(Z-(a-1)) + F_{2a+1}(Z-a) \\ & + F_{2a+1}(Z-(a+1)))] \\ & - [2F_{2a+1}(Z-(a+1))(F_{2a+1}(Z-a))]. \end{aligned} \tag{15}$$

In closing this section, we note that for $k = 1$, we can combine the coefficients in (1) to obtain

$$F_m(n) = \sum_{j=1}^{m} a_j F_m(n-j). \tag{16}$$

Hence, we can solve for $F_m(n)$ in terms of the a_j, where the a_j are arbitrary numbers, that is

$$F_m(0) = 1,\ F_m(1) = a_1,\ F_m(2) = (a_1)^2 + a\ ,\ \text{etc., where } m \geq 1.$$

It might also be noted here that all the numbers, the Fibonacci, Tribonacci, Quadranacci, ..., and Hoganacci numbers, are, respectively, the sums and differences of Fibonacci, Tribonacci, Quadranacci, ..., and Hoganacci *squares*.

III. A Congruence for $F_{m,k}(n)$ and Related Identities

We now return to (1) and consider the function $G(x)^{-k}$. Let

$$y = G(x). \tag{17}$$

Since

$$1/y = 1 + (1-y)/y, \tag{18}$$

we see that

$$1/y = 1 + (a_1x/y) + (a_2x^2/y) + \cdots + (a_mx^m/y). \tag{19}$$

Multiplying (19) by $1/y^k$, we have

$$1/y^{k+1} = 1/y^k + (a_1x/y^{k+1}) + (a_2x\ /y^{k+1}) + \ldots + (a_mx^m/y^{k+1}). \tag{20}$$

Combining the coefficients in (20) leads to

$$F_{m,\,k+1}(n) = \sum_{j=1}^{m}(a_jF_{m,\,k+1}(n-j)) + F_{m,\,k}(n), \quad (n \geq 1). \tag{21}$$

Substitution of (17) into (1) yields

$$1/y^k = \sum_{n=0}^{\infty} F_{m,\,k}(n)x^n, \tag{22}$$

which, after differentiation and multiplying through by x, gives

$$-kx(y^{-k-1}dy/dx) = \sum_{n=1}^{\infty} nF_{m,\,k}(n)x^n. \tag{23}$$

Now, using the values of y in (18) and (1) and combining the coefficients of both sides of (23), we have

$$nF_{m,\,k}(n) = k\sum_{j=1}^{m} ja_jF_{m,\,k+1}(n-j). \tag{24}$$

We observe that (24) is a special case of (12) and (13). Hence, we get the following congruence:

$$F_{m,\,k}(n) \equiv 0 \pmod{k/(n,\,k)}, \text{ for } n \geq k. \tag{25}$$

Multiplying both sides of the equation in (21) by n, we have $nF_{m,\,k}(n)$ in both (21) and (24). Hence, combining (21) with (24) leads to

$$nF_{m,\,k+1}(n) = \sum_{j=1}^{m}(jka_j + na_j)F_{m,\,k+1}(n-j). \tag{26}$$

Replacing k with $k - 1$, we get

$$nF_{m,\,k}(n) = \sum_{j=1}^{m}(j(k-1)a_j + na_j)F_{m,\,k}(n-j), \tag{27}$$

where n, $k \geq 1$ and $m \geq 2$.

IV. A Table of Fibonacci Extensions

We now use equation (21) to make a table of Fibonacci extensions (see [5]) where, for convenience, we let $m = 2$. Of course, we could have considered any other value for m, where $m \geq 3$.

In Table 2, below, we consider the values of $F_{2,\,k}(n)$, where the $a_j = 1$ and $F_{2,\,k}(0) = 1$.

Table 2 was constructed by using the following rule:

> To get the k^{th} element in the n^{th} column, add the k^{th} element in the $(n-1)^{\text{st}}$ column and the k^{th} element in the $(n-2)^{\text{nd}}$ column together with the $(k-1)^{\text{st}}$ element in the n^{th} column. Note that the second row is the Fibonacci numbers.

When $m = 3$, we obtain the Tribonacci numbers. To get the Tribonacci extensions we merely proceed as in Table 2, except that we have one more term, that is, our rule is:

To get the k^{th} element in the n^{th} column of the Tribonacci extensions, we add the k^{th} element in the $(n - 1)^{\text{st}}$ column, the k^{th} element in the $(n - 2)^{\text{nd}}$ column and the k^{th} element in the $(n - 3)^{\text{rd}}$ column together with the $(k - 1)^{\text{st}}$ element in the n^{th} column.

We can construct a table for any $m \geq 4$ in the same way we found the table for $m = 3$ but with added induction.

TABLE 2

	0	1	2	3	4	5	6	...
0	0	0	0	0	0	0	0	...
1	1	1	2	3	5	8	13	...
2	1	2	5	10	20	38	71	...
3	1	3	9	22	51	111	233	...
...	...	...	...	...	...	...	...	...
k	1	k	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...

In order to construct Table 2 for the k^{th} powers, one might think we need to construct k lines, which is a great deal of work. However, this is really not necessary, since by equation (27) it is evident we need only find the numbers in line k.

The following is a table of the generalized Fibonacci numbers. For convenience, we have replaced a_1 with a and a_2 with b, where a and b are arbitrary numbers.

TABLE 3

Values of $F_{2,k}(n, a, b)$

		1	2	3	4
0	0	0	0	0	0
1	1	a	$a^2 + b$	$a^3 + 2ab$	$a^4 + 3a^2b + b^2$
2	1	2a	$3a^2 + 2b$	$4a^3 + 6ab$	$5a^4 + 12a^2b + 3b^2$
3	1	3a	$6a^2 + 2b$	$10a^3 + 12ab$	$15a^4 + 30a^2b + 6b^2$
...	...	...	...	...	...
k	1	ka	...	...	...
...	...	...	...	...	...

Table 3 was constructed using the rule:

To get the k^{th} element in the n^{th} column, we add the product of a multiplied by the k^{th} element in the $(n - 1)^{\text{st}}$ column and the product of b multiplied by the k^{th} element in the $(n - 2)^{\text{nd}}$ column together with the $(k - 1)^{\text{st}}$ element in the n^{th} column.

To obtain the table for $m = 3$ that gives us the generalized Tribonacci numbers, we use the rule:

> To get the k^{th} element in the n^{th} column, we add the product of a_1 multiplied by the k^{th} element in the $(n - 1)^{\text{st}}$ column to the product of a_2 multiplied by the k^{th} element in the $(n - 2)^{\text{nd}}$ column and we add those two products together with the third product of a_3 multiplied by the k^{th} element in the $(n - 3)^{\text{rd}}$ column. We then add the sum of the three products together with the $(k - 1)^{\text{st}}$ element in the n^{th} column.

To obtain the table for $m \geq 4$, we do exactly as we did for $m = 3$ but with added induction.

We conclude this paper by noting that in exactly the way we found Table 2 (with the $a_j = 1$) we may also construct Table 3 with the a_j equal to arbitrary numbers.

Using step-by-step induction, it is easy to show that, by equation (27), we can find any element on line k using only the numbers found on line k and a_j.

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