RECURRING-SEQUENCE TILING

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In this paper we consider the problem of dividing a rectangle into nonoverlapping squares and rectangles using recurring-sequence tiling. The results obtained herein are illustrated with appropriate figures. These results, with the exception of basic introductory material, are believed to be new. There seem to be no such results in the literature.

Among the many generating functions possible, we choose the following:

$$G(x)^{-k} = 1/(1 - x - x^2 - \cdots - x^m)^k \tag{1}$$

(where $m = 2, 3, 4, \ldots$ and $k = 1, 2, 3, \ldots$).

Note that

$$G(x)^{-k} = F_{m,k}(0) + F_{m,k}(1)x + F_{m,k}(2)x^2 + \cdots + F_{m,k}(n)x^n + \cdots \tag{2}$$

[where $F_{m,k}(0) = 1$, for all m and k].

In this paper, we limit k to the value of 1. Therefore, with $k = 1$, for convenience, we can write

$$F_{m,k}(n) = F_m(n).$$

When $m = 2$, the above formulas will result in the well-known Fibonacci numbers. When $m = 3$, one will obtain the Tribonacci numbers.

A tile representing a number in one of these sequences will be a square whose sides are of a length equal to that number. As we examine various versions of equation (2) above, we will attempt to combine tiles so that rectangular regions are formed. When this is not possible, we will identify the gaps left in the almost-rectangular region, and attempt to generalize the sizes of those gaps.

First let $m = 2$, then for $n = 0, 1, 2, \ldots$, we have the Fibonacci numbers and the following consecutive values for $F_2(n)$:

$$F_2(0), F_2(1), F_2(2), \ldots, F_2(n), \ldots$$
$$= 1, 1, 2, 3, 5, 8, 13, 21, \ldots . \tag{3}$$

When tiles are fashioned from these numbers, they may be arranged as in Figure 1. *Note that each tile is a square.* There are, of course, other ways to arrange these tiles. This method of arrangement is simple to follow, and facilitates the arguments below. In this case, as each new tile is added to the region, a full rectangle results.

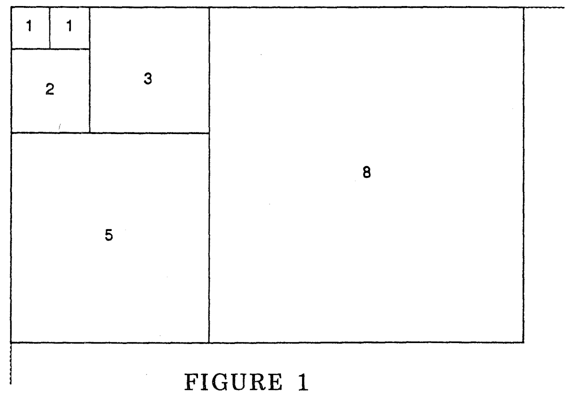

FIGURE 1

When $m = 3$, then, for $n = 0, 1, 2, 3, \ldots$, we have the Tribonacci numbers consisting of the following consecutive values for $F_3(n)$:

$$F_3(0),\ F_3(1),\ F_3(2),\ \ldots,\ F_3(n),\ \ldots$$
$$= 1,\ 1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 81,\ 149,\ 274,\ \ldots\ . \tag{4}$$

Tiles representing these numbers may be arranged as in Figure 2. Again, though distorted because of space, each tile is a square. Note that in this case the addition of a new tile results in a region of irregular shape. To form a full rectangle, smaller rectangles (not necessarily squares) must be added to fill in the "gaps." We shall examine these gaps further below.

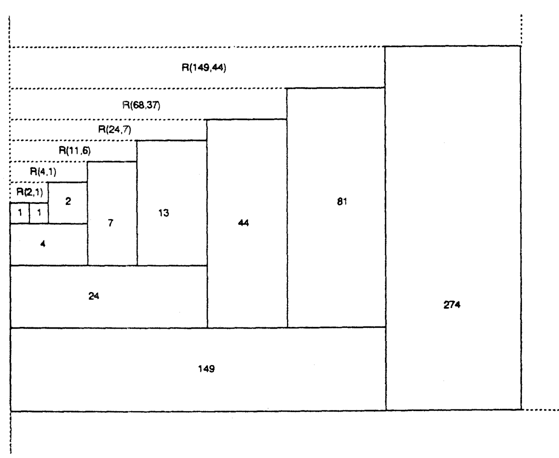

FIGURE 2

We will later continue in this way for larger values of m, and step by step, and with added induction we will develop a systematic way of placing tiles using the recurring sequence in equation (2) above.

Let us now examine the gaps in the rectangle we are attempting to tile. Refer to the upper left corner of Figure 2 to see these gaps. Our approach will be to find a recurrence relation which might be used to construct smaller rectangles, which will in turn fill in the gaps. We shall label these smaller rectangles $R_m(x, y)$, where x and y are the horizontal and vertical components of the rectangles, respectively.

In Table 3 below, for $m = 3$, we list the consecutive values of the horizontal component x (where $x = 2, 4, 11, 24, 68, 149, \ldots$) in the consecutive rectangles $R_m(x, y)$.

TABLE 3

Horizontal Components of Rectangles, for $m = 3$

Rectangle	x	Formula
R(2,1)	2	$F_3(0) + F_3(1)$
R(4,1)	4	$F_3(3)$
R(11,6)	11	$F_3(3) + F_3(4)$
R(24,7)	24	$F_3(6)$
R(68,37)	68	$F_3(6) + F_3(7)$
R(149,44)	149	$F_3(9)$
...	...	...

Now, using Table 3 and by induction, we obtain the following general values of x in $R_3(x, y)$:

$$x = F_3(3n) + F_3(3n + 1) \quad \text{or} \quad x = F_3(3n + 3), \text{ where } n \geq 0. \tag{5}$$

In Table 4 below, for $m = 3$, we list the consecutive values of the vertical component y (for $y = 1, 1, 6, 7, 37, 44, \ldots$) of each $R_3(x, y)$ from Figure 2:

TABLE 4

Vertical Components of Rectangles, for $m = 3$

Rectangle	y	Formula
R(2,1)	1	$F_3(0)$
R(4,1)	1	$F_3(1)$
R(11,6)	6	$F_3(2) + F_3(3)$
R(24,7)	7	$F_3(4)$
R(68,37)	37	$F_3(5) + F_3(6)$
R(149,44)	44	$F_3(7)$
...	...	...

Now, using Table 4 step by step and with added induction, we obtain the following values of y in $R_3(x, y)$:

$$y = F_3(0) = 1 \quad \text{or} \quad y = F_3(3n + 1) \quad \text{or}$$
$$F_3(3n + 2) + F_3(3n + 3), \text{ where } n \geq 0. \tag{6}$$

Combining (5) and (6) above, we observe that the general form of each gap-filling rectangle $R_3(x, y)$ may be written as

$$R_3(2, 1) \quad \text{or} \quad R_3(F_3(3n + 3), F_3(3n + 1)) \quad \text{or}$$
$$R_3(F_3(3n + 3) + F_3(3n + 4), F_3(3n + 2) + F_3(3n + 3)), \text{ where } n \geq 0. \tag{7}$$

We now consider the rectangles that we are attempting to tile. The notation we use will be $A_m(x, y)$, where once again x and y are the horizontal and vertical lengths of the tiled rectangle.

For $m = 3$, we have $A_3(x, y)$. As we add the tiles one by one, certain of the rectangles $A_3(x, y)$ tiled, in order of their construction, are shown in Table 5. Also shown are the components of each tiled rectangle, with each square tile followed by an S.

TABLE 5

Construction of Tiled Rectangles, for $m = 3$

Tiled Rectangle	Component Squares and Rectangles
$A_3(2,1)$	1S + 1S
$A_3(4,6)$	$A_3(2,1)$ + 2S + 4S + $R_3(2,1)$
$A_3(11,7)$	$A_3(4,6)$ + 7S + $R_3(4,1)$
$A_3(24,37)$	$A_3(11,7)$ + 13S + 24S + $R_3(11,6)$
$A_3(68,44)$	$A_3(24,37)$ + 44S + $R_3(24,7)$
...	...

Continuing in this way, by induction, we conclude that the general formulas for the areas $A_3(x, y)$ are

$$A_3(F_3(3n) + F_3(3n + 1), F_3(3n + 1)) \quad \text{or}$$
$$A_3(F_3(3n + 3), F_3(3n + 2) + F_3(3n + 3)), \text{ where } n = 0, 1, 2, \ldots . \tag{8}$$

Now, let $m = 4$ in equation (2). Then, for $n = 0, 1, 2, \ldots$, we have the following consecutive values for $F_4(n)$:

$$F_4(0), F_4(1), F_4(2), \ldots, F_4(n), \ldots$$
$$= 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, \ldots . \tag{9}$$

An arrangement of the tiles corresponding to these values is shown in Figure 6. Again, note that the arrangement of square tiles results in an irregular shape. Once again, "filler" rectangles must be generated to fill in the gaps, to construct a fully tiled rectangle.

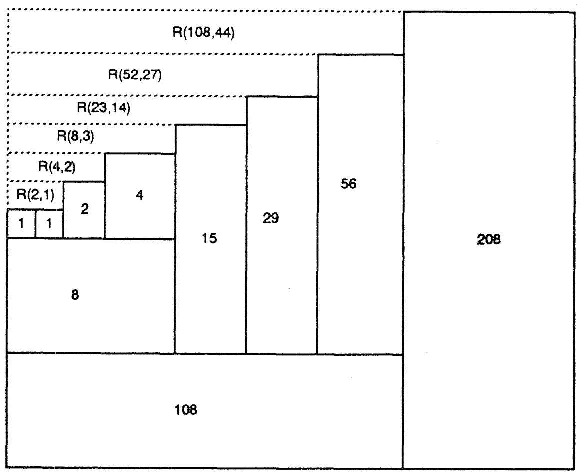

FIGURE 6

In Table 7 below, for $m = 4$, we list the consecutive values of the horizontal component x (where $x = 2, 4, 8, 23, 52, 108, 316, 717, 1490, \ldots$) in the consecutive rectangles $R_4(x, y)$.

TABLE 7

Horizontal Components of Rectangles, for $m = 4$

Rectangle	x	Formula
R(2,1)	2	$F_4(0) + F_4(1)$
R(4,2)	4	$F_4(0) + F_4(1) + F_4(2)$
R(8,3)	8	$F_4(4)$
R(23,14)	23	$F_4(4) + F_4(5)$
R(52,27)	52	$F_4(4) + F_4(5) + F_4(6)$
R(108,44)	108	$F_4(8)$
...	...	...

Now, using Table 7 with added induction, we find the following general values of x in $R_4(x, y)$:

$$x = F_4(4n) + F_4(4n + 1) \quad \text{or}$$

$$x = F_4(4n) + F_4(4n + 1) + F_4(4n + 2) \quad \text{or}$$

$$x = F_4(4n + 4), \text{ where } n \geq 0. \tag{10}$$

In Table 8 below, for $m = 4$, we list the consecutive values of the vertical component y ($y = 1, 2, 3, 14, 27, 44, 193, 372, 609, \ldots$) of each $R_4(x, y)$:

TABLE 8

Vertical Components of Rectangles, for $m = 4$

Rectangle	y	Formula
R(2,1)	1	$F_4(1)$
R(4,2)	2	$F_4(2)$
R(8,3)	3	$F_4(1) + F_4(2)$
R(23,14)	14	$F_4(2) + F_4(3) + F_4(4)$
R(52,27)	27	$F_4(3) + F_4(4) + F_4(5)$
R(108,44)	44	$F_4(5) + F_4(6)$
...	...	...

Now, using Table 8 step by step and with added induction, we find the following general values of y in $R_4(x, y)$:

$$y = F_4(1) = 1 \quad \text{or} \quad y = F_4(2) \quad \text{or} \tag{11}$$
$$y = F_4(4n + 1) + F_4(4n + 2) \quad \text{or}$$
$$y = F_4(4n + 2) + F_4(4n + 3) + F_4(4n + 4) \quad \text{or}$$
$$y = F_4(4n + 3) + F_4(4n + 4) + F_4(4n + 5), \text{ where } n = 0, 1, 2, \ldots .$$

Combining equations (10) and (11) above, we observe that the general formulas of the $R_4(x, y)$ may be written as

$$R_4(2, 1), \quad R_4(4, 2) \tag{12}$$
$$R_4(F_4(4n + 4), F_4(4n + 1) + F_4(4n + 2)),$$
$$R_4(F_4(4n + 4) + F_4(4n + 5), F_4(4n + 2) + F_4(4n + 3) + F_4(4n + 4)),$$
$$R_4(F_4(4n + 4) + F_4(4n + 5) + F_4(4n + 6),$$
$$F_4(4n + 3) + F_4(4n + 4) + F_4(4n + 5)), \text{ where } n \geq 0.$$

We now consider selected tiled rectangles $A_4(x, y)$ in the order of their construction:

TABLE 9

Construction of Tiled Rectangles, for $m = 4$

Tiled Rectangle	Component Squares and Rectangles
$A_4(2,1)$	1S + 1S
$A_4(4,2)$	$A_4(2,1)$ + 2S + $R_4(2,1)$
$A_4(8,12)$	$A_4(4,2)$ + 4S + 8S + $R_4(4,2)$
$A_4(23,15)$	$A_4(8,12)$ + 15S + $R_4(8,3)$
$A_4(52,29)$	$A_4(23,15)$ + 29S + $R_4(23,14)$
$A_4(108,164)$	$A_4(52,29)$ + 56S + 108S + $R_4(52,27)$
...	...

Continuing in this way, we conclude that the general formulas for the areas $A_4(x, y)$ are

$$A_4(F_4(4n) + F_4(4n + 1), F_4(4n + 1)),$$
$$A_4(F_4(4n) + F_4(4n + 1) + F_4(4n + 2), F_4(4n + 2)),$$
$$A_4(F_4(4n + 4), F_4(4n + 3) + F_4(4n + 4)), \text{ where } n \geq 0. \qquad (13)$$

Now, by induction, we tile the functions in equation (2) for all $m = 5, 6, 7, \ldots$. We are concerned with the sequence of values for $F_m(n)$:

$$F_m(0), F_m(1), F_m(2), \ldots, F_m(n), \ldots$$
$$= 1, 1, 2, 4, 8, 16, \ldots, 2^{m-1}, 2^m - 1, 2^{m+1} - 3,$$
$$2^{m+2} - 8, 2^{m+3} - 20, \ldots, \text{ where } n \geq 0. \qquad (14)$$

An arrangement of the tiles corresponding to these values is shown in Figure 10.

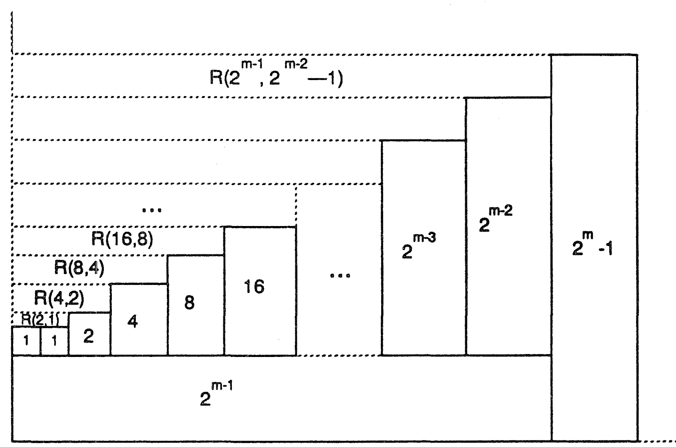

FIGURE 10

Now, by induction, we may systematically generalize the values of the horizontal component x in each rectangle $F_m(x, y)$:

For $n = 0$,

$$F_m(m \cdot 0 + m)$$
$$F_m(m \cdot 0 + m) + F_m(m \cdot 0 + m + 1)$$
$$F_m(m \cdot 0 + m) + F_m(m \cdot 0 + m + 1) + F_m(m \cdot 0 + m + 2)$$
$$\vdots$$
$$F_m(m \cdot 0 + m) + F_m(m \cdot 0 + m + 1) + F_m(m \cdot 0 + m + 2) + \cdots + F_m(m \cdot 0 + 2m - 2).$$

For $n = 1$,

$$F_m(m \cdot 1 + m)$$
$$F_m(m \cdot 1 + m) + F_m(m \cdot 1 + m + 1)$$
$$F_m(m \cdot 1 + m) + F_m(m \cdot 1 + m + 1) + F_m(m \cdot 1 + m + 2)$$
$$\vdots$$
$$F_m(m \cdot 1 + m) + F_m(m \cdot 1 + m + 1) + F_m(m \cdot 1 + m + 2) + \cdots + F_m(m \cdot 1 + 2m - 2).$$

Then, in general, for n (where $n \geq 0$),

$$F_m(m \cdot n + m) \tag{15}$$
$$F_m(m \cdot n + m) + F_m(m \cdot n + m + 1)$$
$$F_m(m \cdot n + m) + F_m(m \cdot n + m + 1) + F_m(m \cdot n + m + 2)$$
$$\vdots$$
$$F_m(m \cdot n + m) + F_m(m \cdot n + m + 1) + F_m(m \cdot n + m + 2) + \cdots + F_m(m \cdot n + 2m - 2).$$

Now, step by step and with added induction, we obtain the generalized values of the vertical component y in each rectangle $R_m(x, y)$:

For $n = 0$,

$$F_m(m \cdot 0 + 1) + F_m(m \cdot 0 + 2) + \cdots + F_m(m \cdot 0 + m - 2)$$
$$F_m(m \cdot 0 + 2) + F_m(m \cdot 0 + 3) + \cdots + F_m(m \cdot 0 + m)$$
$$F_m(m \cdot 0 + 3) + F_m(m \cdot 0 + 4) + \cdots + F_m(m \cdot 0 + m + 1)$$
$$\vdots$$
$$F_m(m \cdot 0 + m - 1) + F_m(m \cdot 0 + m) + \cdots + F_m(m \cdot 0 + 2m - 3).$$

For $n = 1$,

$$F_m(m \cdot 1 + 1) + F_m(m \cdot 1 + 2) + \cdots + F_m(m \cdot 1 + m - 2)$$
$$F_m(m \cdot 1 + 2) + F_m(m \cdot 1 + 3) + \cdots + F_m(m \cdot 1 + m)$$
$$F_m(m \cdot 1 + 3) + F_m(m \cdot 1 + 4) + \cdots + F_m(m \cdot 1 + m + 1)$$
$$\vdots$$
$$F_m(m \cdot 1 + m - 1) + F_m(m \cdot 1 + m) + \cdots + F_m(m \cdot 1 + 2m - 3).$$

For n is general, where $n = 0, 1, 2, \ldots,$

$$F_m(m \cdot n + 1) + F_m(m \cdot n + 2) + \cdots + F_m(m \cdot n + m - 2) \tag{16}$$
$$F_m(m \cdot n + 2) + F_m(m \cdot n + 3) + \cdots + F_m(m \cdot n + m)$$
$$F_m(m \cdot n + 3) + F_m(m \cdot n + 4) + \cdots + F_m(m \cdot n + m + 1)$$
$$\vdots$$
$$F_m(m \cdot n + m - 1) + F_m(m \cdot n + m) + \cdots + F_m(m \cdot n + 2m - 3).$$

One should note that

$$F_m(m \cdot 0 + m) = 2^{m-1}$$

and

$$F_m(m \cdot 0 + 1) + F_m(m \cdot 0 + 2) + \cdots + F_m(m \cdot 0 + m - 2) = 2^{m-2} - 1.$$

Combining the generalized formulas for the x and y components of each rectangle, equations (15) and (16) above, we observe that the general formulas of each rectangle $R_m(x, y)$ may be written as

$$R_m(2, 1) \tag{17}$$
$$R_m(4, 2)$$
$$R_m(8, 4)$$
$$R_m(16, 8)$$
$$\vdots$$
$$R_m(2^{m-2}, 2^{m-3})$$

$$R_m(F_m(m\cdot 0+m),\ F_m(m\cdot 0+1)+F_m(m\cdot 0+2)+\cdots+F_m(m\cdot 0+m-2))$$

$$R_m(F_m(m\cdot 0+m)+F_m(m\cdot 0+m+1),$$
$$F_m(m\cdot 0+2)+F_m(m\cdot 0+3)+\cdots+F_m(m\cdot 0+m))$$

$$R_m(F_m(m\cdot 0+m)+F_m(m\cdot 0+m+1)+F_m(m\cdot 0+m+2),$$
$$F_m(m\cdot 0+3)+F_m(m\cdot 0+4)+\cdots+F_m(m\cdot 0+m+1))$$

$$\vdots$$

$$R_m(F_m(m\cdot 0+m)+F_m(m\cdot 0+m+1)+F_m(m\cdot 0+m+2)$$
$$+\cdots+F_m(m\cdot 0+2m-2),\ F_m(m\cdot 0+m-1)+F_m(m\cdot 0+m)$$
$$+\cdots+F_m(m\cdot 0+2m-3))$$

$$R_m(F_m(m\cdot 1+m),\ F_m(m\cdot 1+1)+F_m(m\cdot 1+2)+\cdots+F_m(m\cdot 1+m-2))$$

$$R_m(F_m(m\cdot 1+m)+F_m(m\cdot 1+m+1)$$
$$F_m(m\cdot 1+2)+F_m(m\cdot 1+3)+\cdots+F_m(m\cdot 1+m))$$

$$R_m(F_m(m\cdot 1+m)+F_m(m\cdot 1+m+1)+F_m(m\cdot 1+m+2),$$
$$F_m(m\cdot 1+3)+F_m(m\cdot 1+4)+\cdots+F_m(m\cdot 1+m+1))$$

$$\vdots$$

$$R_m(F\ \ (m\cdot 1+m)+F_m(m\cdot 1+m+1)+F_m(m\cdot 1+m+2)$$
$$+\cdots+F_m(m\cdot 1+2m-2),\ F_m(m\cdot 1+m-1)+F_m(m\cdot 1+m)$$
$$+\cdots+F_m(m\cdot 1+2m-3))$$

$$R_m(F_m(m\cdot n+m),\ F_m(m\cdot n+1)+F_m(m\cdot n+2)+\cdots+F_m(m\cdot n+m-2))$$

$$R_m(F_m(m\cdot n+m)+F_m(m\cdot n+m+1),$$
$$F_m(m\cdot n+2)+F_m(m\cdot n+3)+\cdots+F_m(m\cdot n+m))$$

$$R_m(F_m(m\cdot n+m)+F_m(m\cdot n+m+1)+F_m(m\cdot n+m+2),$$
$$F_m(m\cdot n+3)+F_m(m\cdot n+4)+\cdots+F_m(m\cdot n+m+1))$$

$$\vdots$$

$$R_m(F_m(m\cdot n+m)+F_m(m\cdot n+m+1)+F_m(m\cdot n+m+2)$$
$$+\cdots+F_m(m\cdot n+2m-2),$$
$$F_m(m\cdot n+m-1)+F_m(m\cdot n+m)+\cdots+F_m(m\cdot n+2m-3)),$$

where $n = 0, 1, 2, \ldots$.

We now consider the tiled rectangles $A_m(x, y)$ in order of their construction:

$$A_m(F_m(m\cdot 0)+F_m(m\cdot 0+1),\ F_m(m\cdot 0+1)) \tag{18}$$

$$A_m(F_m(m\cdot 0)+F_m(m\cdot 0+1)+F_m(m\cdot 0+2),\ F_m(m\cdot 0+2))$$

$$A_m(F_m(m\cdot 0)+F_m(m\cdot 0+1)+F_m(m\cdot 0+2)+F_m(m\cdot 0+3),\ F_m(m\cdot 0+3))$$

$$\vdots$$

$$A_m(F_m(m\cdot 0)+F_m(m\cdot 0+1)+F_m(m\cdot 0+2)$$
$$+\cdots+F_m(m\cdot 0+m-2),\ F_m(m\cdot 0+m-2))$$

$$A_m(F_m(m\cdot 0+m),\ F_m(m\cdot 0+m-1)+F_m(m\cdot 0+m))$$

$$A_m(F_m(m\cdot 1)+F_m(m\cdot 1+1),\ F_m(m\cdot 1+1))$$

$$A_m(F_m(m\cdot 1)+F_m(m\cdot 1+1)+F_m(m\cdot 1+2),\ F_m(m\cdot 1+2))$$

$$A_m(F\ (m\cdot 1) + F_m(m\cdot 1+1) + F_m(m\cdot 1+2) + F_m(m\cdot 1+3),\ F_m(m\cdot 1+3))$$

$$\vdots$$

$$A_m(F_m(m\cdot 1) + F_m(m\cdot 1+1) + F_m(m\cdot 1+2)$$
$$+ \cdots + F_m(m\cdot 1+m-2),\ F_m(m\cdot 1+m-2))$$

$$A_m(F_m(m\cdot 1+m),\ F_m(m\cdot 1+m-1) + F_m(m\cdot 1+m))$$

$$A_m(F_m(mn)+F_m(mn+1),\ F_m(mn+1))$$

$$A_m(F_m(mn) + F_m(mn+1) + F_m(mn+2),\ F_m(mn+2))$$

$$A_m(F_m(mn)+F_m(mn+1) + F_m(mn+2) + F_m(mn+3),\ F_m(mn+3))$$

$$\vdots$$

$$A_m(F_m(mn) + F_m(mn+1) + F_m(mn+2)$$
$$+ \cdots + F_m(mn+m-2),\ F_m(mn+m-2))$$

$$A_m(F_m(mn+m),\ F_m(mn+m-1) + F_m(mn+m)),$$

where $n = 0, 1, 2, \ldots$.

This establishes the recurring sequences for the tiling with $k = 1$ and $m = 2, 3, 4, \ldots$ for equation (1), using the construction of Figure 10. We intend to generalize this procedure for $k > 1$ in later work.

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