THE ARITHMETIC OF POWERS AND ROOTS IN $Gl_2(C)$ AND $Sl_2(C)$

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(Submitted June 1986)

Introduction

Let A be in $\mathbf{Sl}_2(\mathbf{C})$ ("The special linear group of degree 2 over $\mathbf{C}$"; see [5]) and let n be a positive integer. Let us look at all B's in $\mathbf{Sl}_2(\mathbf{C})$ for which $B^n = A$.

If $\chi = \operatorname{Tr} A \neq \pm 2$, then A is diagonalizable since it has two different eigenvalues, namely, $(\chi \pm \sqrt{\chi^2 - 4})/2$, and it is trivial to compute all n^{th} roots of A.

If A is the identity matrix and if δ is an eigenvalue of some n^{th} root B of A, then, unless $\delta = \pm 1$, the other eigenvalue of B is different (as it is $1/\delta$, the determinant being 1) and therefore B is diagonalizable, that is, B is a conjugate of $\begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix}$, with δ an n^{th} root of 1; note that when $\delta = \pm 1$, it is easy to check that B is $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The case $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is similar.

Finally, if $\chi = \pm 2$, but $A \neq \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the problem is slightly more difficult; it turns out that there are either 0, 1, or 2 n^{th} root(s) in $\mathbf{Sl}_2(\mathbf{C})$, depending on n and A.

If $A \in \mathbf{Gl}_2(\mathbf{C})$ ("The general linear group of degree 2 over $\mathbf{C}$"; see [5]) is not a multiple of the identity, then A has exactly n n^{th} roots. If A is a multiple of the identity, then A has infinitely many n^{th} roots for any n.

Although we will compute roots in $\mathbf{Sl}_2(\mathbf{C})$ and $\mathbf{Gl}_2(\mathbf{C})$, the immediate purpose of this paper *is not* to compute roots in these groups. Our purpose is to give a *nonlinear-algebra* approach to computing roots which rests on the arithmetic involved in computing the powers of an element of $\mathbf{Sl}_2(\mathbf{C})$ or $\mathbf{Gl}_2(\mathbf{C})$. Computing these powers involves a finite number of multiplications and additions; this gives rise to polynomials and the arithmetic of these polynomials yields another method to compute roots in $\mathbf{Sl}_2(\mathbf{C})$ *without any linear-algebra concept*. We obtain a complete description of these roots in this way, with transcendental functions in expressions not naturally given by the linear-algebra approach [see, e.g., (1.14-C)]. We will explore this arithmetic and see how it connects most naturally with Chebyshev's polynomials. It also yields a *natural* meaning to *arbitrary complex powers* in $\mathbf{Sl}_2(\mathbf{C})$ and $\mathbf{Gl}_2(\mathbf{C})$, and we obtain an explicit formula allowing computations of A for any n in a time which theoretically does not depend on n [see (1.6), (1.8), and (2.1)]. As far as computing roots is concerned, the arithmetic of these polynomials gives an elegant nonlinear-algebra solution which solves the problem of extracting roots in all cases in the same way, be the matrix diagonalizable or not.

Computing roots of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is achieved first through computing roots of

$$A/\sqrt{ad - bc} \text{ [which is trivially in } \mathbf{Sl}_2(\mathbf{C})\text{]};$$

therefore, we first study the arithmetic of powers and roots in $\mathbf{Sl}_2(\mathbf{C})$. It rests on two families of polynomials; if χ and χ_n are, respectively, the traces of A and

$$A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

then χ_n is a polynomial in χ which depends only on n and not on A. In addition, there is a polynomial P_n which gives the values of b_n and c_n through

$$b_n = bP_n(\chi) \quad \text{and} \quad c_n = cP_n(\chi).$$

These polynomials are deeply related to Chebyshev's polynomials and a full description of their zeros yields a full description of all roots of any element of $\mathbf{Sl}_2(\mathbf{C})$. The P_n's, which appear naturally in our problem have been considered more or less directly in some other contexts (see [1], [3], and [4]).

The P_n's have received much attention, but as far as we know the χ_n's have received little; the computation of roots in $\mathbf{Sl}_2(\mathbf{C})$ has also received little attention because in most practical cases there is an obvious linear-algebra solution (which however masks the arithmetic behind the calculations). As far as the raw computation of roots is concerned, we found a vague and partial answer in [6] which triggered our investigation, and an exercise in [2] coming from [7] which concerns the sole case when A is hermitian and $n = 2$. We are thankful to Professor G. Bergum for bringing to our attention references [3] and [4] regarding the P_n's.

Powers and Roots in $\mathbf{Sl}_2(\mathbf{C})$

The starting point of this paper is the following family of polynomials: for each $n \in \mathbf{Z}$, we define a polynomial P_n by

$$\text{(a)} \quad P_0(t) = 0 \text{ and } P_1(t) = 1; \quad \text{(b)} \quad P_{n+1}(t) = tP_n(t) - P_{n-1}(t) \tag{1.1}$$

These polynomials have the easily verified properties:

$$\text{a)} \quad P_n(\pm 2) = n(\pm 1)^{n+1}; \quad \text{b)} \quad P_{-n} = -P_n. \tag{1.2}$$

Their roots are studied in [3] and [4], where $P_n = A_{2n}$ in their notation. The following proposition, the proof of which is an easy induction on n using $ad - bc = 1$ [this matrix is in $\mathbf{Sl}_2(\mathbf{C})$], ignited our interest in this family of polynomials; we lately discovered a more general version of it in [1], but we state in Proposition 1 just the particular case we need.

Proposition 1: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathbf{Sl}_2(\mathbf{C})$ and let us set $\chi = a + d$. Then, for each $n \in \mathbf{Z}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} aP_n(\chi) - P_{n-1}(\chi) & bP_n(\chi) \\ cP_n(\chi) & dP_n(\chi) - P_{n-1}(\chi) \end{pmatrix} \tag{1.3}$$

Corollary: For $A \in \mathbf{Sl}_2(\mathbf{C})$ and $n \in \mathbf{Z}$, if χ and χ_n are, respectively, the trace of A and A^n, then

$$\chi_n = P_{n+1}(\chi) - P_{n-1}(\chi). \tag{1.4}$$

Using (1.4) as a motivation, we introduce yet another family of polynomials: for each $n \in \mathbf{Z}$, we set

$$\chi_n(t) = P_{n+1}(t) - P_{n-1}(t);$$

each χ_n is a polynomial of degree $|n|$; moreover $\chi_n = \chi_{-n}$, as is easily checked from $P_n = -P_{-n}$. The table in the Appendix shows these polynomials for all values of n in the range $2 \le n \le 20$.

We shall need the zeros of all polynomials of the form $\chi_n - \xi$, with $\xi \in \mathbf{C}$. Fortunately, these zeros are easy to describe and, surprisingly, this result seems to be new.

Proposition 2: Let $n > 0$ and ξ be an arbitrary complex number, and let us set $\rho = \xi/2$. Then the n complex numbers $\xi_0, \xi_1, \ldots, \xi_{n-1}$ defined by

$$\xi_k = 2\cosh\left(\frac{\operatorname{argcosh}\rho + 2k\pi i}{n}\right) = 2\cos\left(\frac{\arccos\rho + 2k\pi}{n}\right) \tag{1.5}$$

are the zeros of $\chi_n - \xi$ $(k = 0, \ldots, n-1)$.

Proof: If T_n is the n^{th} Chebyshev polynomial of the first kind (see [8] or [9]), then one easily proves that the T_n's are defined in terms of the P_n's by

$$2T_n(t) = P_{n+1}(2t) - P_{n-1}(2t). \tag{1.6}$$

Since we look for the solutions of

$$\frac{\chi_n}{2} = \frac{\xi}{2}, \text{ or equivalently of } \frac{P_{n+1}(x) - P_{n-1}(x)}{2} = \rho,$$

when we set $x = 2s$, the problem reduces, using (1.6), to solving $T_n(s) = \rho$; using the identities $T_n(\cos\theta) = \cos(n\theta)$ and $T_n(\cosh\theta) = \cosh(n\theta)$, we see that $T_n(s) = \rho$ has n solutions, which are given by

$$s_k = \cosh\left(\frac{\operatorname{argcosh}\rho + 2k\pi i}{n}\right) = \cos\left(\frac{\arccos\rho + 2k\pi}{n}\right), \tag{1.7}$$

where $k = 0, \ldots, n-1$ (simply write $T_n(s)$ as $T_n(\cos\arccos s) = \rho \ldots$]. These solutions yield the solutions $\xi_k = 2s_k$. Q.E.D.

Remark: It follows from (1.6) that the value of the n^{th} Chebyshev polynomial at any complex number s is the half-trace of A^n, where A is *any* element in $\mathbf{Sl}_2(\mathbf{C})$ with half-trace s. Therefore, if s_n is the half-trace of the n^{th} power of an element of A in $\mathbf{Sl}_2(\mathbf{C})$ with half-trace s, we have

$$s_n = \cosh(n \operatorname{argcosh} s) = \cos(n \arccos s).$$

This is an easy exercise in linear algebra since given A there exists an invertible matrix X such that

$$XAX^{-1} = \begin{pmatrix} \delta & * \\ 0 & 1/\delta \end{pmatrix}.$$

Because the trace is invariant under conjugation, we have

$$s = \cosh(\ln\delta) \quad \text{and} \quad s_n = \cosh(\ln\delta^n) = \cosh(n \operatorname{argcosh} s).$$

Next we need an explicit description of the zeros of the P_n's. These are known (see [3] and [4], where A_{2n} in their notation is our P_n) but our proof is simpler and yields an explicit expression for the values of the P_n's [see (1.8) below] which is used in proving Proposition 5.

Proposition 3: For each integer n, the zeros of P_n are

$$S_k = 2\cos(k\pi/|n|), \quad (k = 1, \ldots, |n| - 1).$$

In particular, they are all real and distinct.

Proof: In view of (1.2)-(b), we will suppose, in full generality, that $n > 0$. Using the easily proved identity

$$(s^2 - 1)P_n(2s) = T_{n+1}(s) - sT_n(s),$$

which defines the P_n's in terms of the T_n's, and the trivial identities (see [8])

$$s = \cosh(\operatorname{argcosh} s) \quad \text{and} \quad T_k(\cosh x) = \cosh(kx),$$

we have

$$(s^2 - 1)P_n(2s) = \sinh(\operatorname{argcosh} s)\sinh(n \operatorname{argcosh} s).$$

Upon writing $s = \cosh(\operatorname{argcosh} s)$, using the standard identities for hyperbolic functions and using the relation (1.2)-(a) to take care of the case $s = \pm 1$, we obtain the following explicit formula for $P_n(2s)$, which one will observe gives the value of P_n without any iteration, and hence of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n$$

without iteration:

$$P_n(2s) = \begin{cases} ns^{n+1} & \text{if } s = \pm 1; \\ \dfrac{\sinh(n \operatorname{argcosh} s)}{\sinh(\operatorname{argcosh} s)} & \text{otherwise.} \end{cases} \tag{1.8}$$

[Note that, with $s \neq \pm 1$, the denominator of the lower part of (1.8) cannot be 0, and that the value of the quotient does not depend on which value is chosen for argcosh s.] It follows from (1.8) that the solutions of $P_n(2s) = 0$ are the values of s for which (n argcosh s) is a multiple of πi; these values are given by

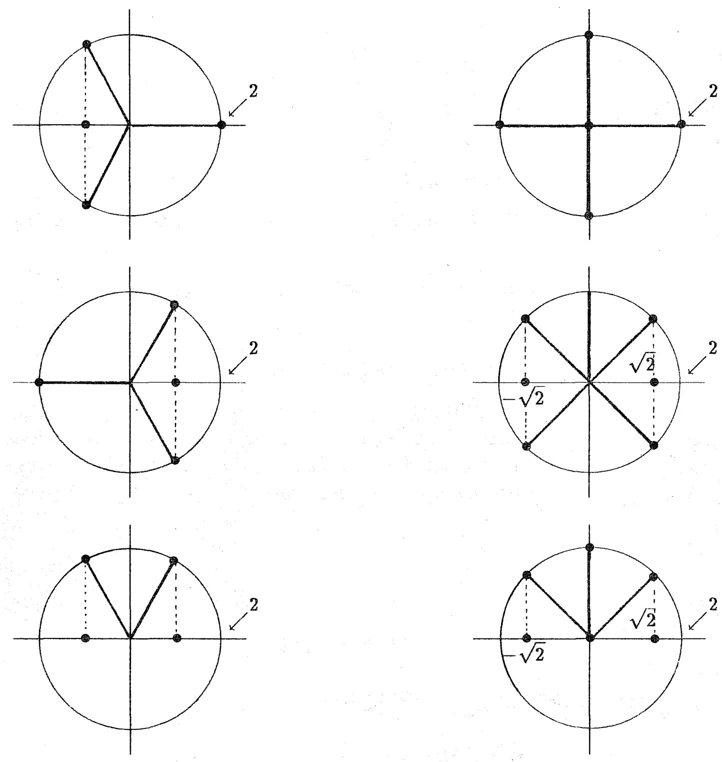
$$s_k = \cos\left(\frac{k\pi}{n}\right), \quad (k = 1, \ldots, n - 1),$$

whence the result

$$S_k = 2s_k = 2\cos\left(\frac{k\pi}{n}\right), \quad (k = 1, \ldots, n - 1). \quad \text{Q.E.D.}$$

It is interesting here to compare the zeros of $\chi_n \pm 2$ with those of P_n. From (1.5):

$$\left\{\begin{array}{ll} \text{The zeros of } \chi_n - 2: & \xi_0 = 2\cos 0,\ \xi_1 = 2\cos\frac{2\pi}{n},\ \ldots,\ \xi_{n-1} = 2\cos\frac{2(n-1)\pi}{n} \\ \text{The zeros of } \chi_n + 2: & \xi_0 = 2\cos\frac{\pi}{n},\ \xi_1 = 2\cos\frac{3\pi}{n},\ \ldots,\ \xi_{n-1} = 2\cos\frac{(2n-1)\pi}{n} \\ \text{The zeros of } P_n: & S_1 = 2\cos\frac{\pi}{n},\ S_2 = \cos\frac{2\pi}{n},\ \ldots,\ S_{n-1} = \cos\frac{(n-1)\pi}{n} \end{array}\right\}.$$



Zeros	Zeros
$\chi_3 - 2: \ \xi_0 = 2,\ \xi_1 = -1,\ \xi_2 = -1$	$\chi_4 - 2: \ \xi_0 = 2,\ \xi_1 = 0,\ \xi_2 = -2,\ \xi_3 = 0$
$\chi_3 + 2: \ \xi_1 = 1,\ \xi_2 = -2,\ \xi_3 = 1$	$\chi_4 + 2: \ \xi_0 = \sqrt{2},\ \xi_1 = -\sqrt{2},\ \xi_2 = -\sqrt{2},\ \xi_3 = \sqrt{2}$
$P_3: \ S_1 = 1,\ S_2 = -1$	$P_4: \ S_1 = \sqrt{2},\ S_2 = 0,\ S_3 = -\sqrt{2}$

FIGURE 1

The zeros of $\chi_n + 2$, $\chi_n - 2$, and P_n

Figure 1 shows the cases $n = 3$ and $n = 4$ and illustrates the essential content of Corollary 1 below; for convenience, let us call the zeros of $\chi_n + 2$ and $\chi_n - 2$ "small zeros" when they are *strictly less* than 2 in absolute value. Then, we clearly have the following:

Corollary 1:

1. The small zeros of $\chi_n + 2$ and $\chi_n - 2$ are each of multiplicity 2.
2. The small zeros of $\chi_n + 2$ and the small zeros of $\chi_n - 2$ form two disjoint subsets, the union of which is the set of zeros of P_n.

Corollary 2: For any $n \in \mathbb{Z}$ and for any $\xi \neq \pm 2$, P_n does not vanish on a zero of $\chi_n - \xi$.

Corollary 1 states something on the values of P_n at the zeros of $\chi_n \pm 2$, and Corollary 2 on the values of P_n at the zeros of $\chi_n - \xi$, with $\xi \neq \pm 2$. If we agree to say that a function f separates n points $z_1, \ldots, z_n$ when f takes n different values on $\{z_1, \ldots, z_n\}$, then Proposition 4 below completes the information of Corollaries 1 and 2. This proposition will be responsible for the fact that the nonmultiple of the identity in $\mathrm{Gl}_2(\mathbf{C})$ has *exactly n distinct* n^{th} roots.

Proposition 4: For all $n \in \mathbb{Z}$ and all $\xi \neq \pm 2$, P_n separates the $|n|$ zeros of $\chi_n - \xi$.

Proof: Since $P_{-n} = -P_n$ and $\chi_{-n} = \chi_n$, we may suppose, in all generality, that $n \geq 0$. The cases $n = 0$ and $n = 1$ are vacuously true because χ_0 and χ_1 have, respectively, 0 and 1 zero [recall that $\chi_0(t) = 2$ and $\chi_1(t) = t$]. Therefore, we suppose that $n \geq 2$.

In order to consider the value of P_n on each of the zeros of $\chi_n - \xi$, let us set

$$a + bi = \frac{\operatorname{argcosh} \xi/2}{n}.$$

Saying that $\xi \neq \pm 2$ means that $a + bi$ is not a multiple of $\pi i/n$. The roots of $\chi_n - \xi$ are, after (1.5),

$$\xi_k = 2 \cosh\left(a + bi + \frac{2k\pi i}{n}\right) \quad (k = 0, \ldots, n - 1),$$

and therefore,

$$P_n(\xi_k) = P_n\left(2 \cosh\left(a + bi + \frac{2k\pi i}{n}\right)\right).$$

Now,

$$\cosh\left(a + bi + \frac{2k\pi i}{n}\right) \neq \pm 1,$$

since the contrary would imply that $a + bi$ is a multiple of $\pi i/n$. It follows from (1.8) that, for $r = 0, \ldots, n - 1$,

$$P_n(\xi_r) = \frac{\sinh n(a + bi)}{\sinh a \cos\left(b + \frac{2\pi r}{n}\right) + i \cosh a \sin\left(b + \frac{2\pi r}{n}\right)}. \tag{1.9}$$

$\left[\text{The denominator is the expansion of } \sinh\left(a + bi + \frac{2\pi r i}{n}\right).\right]$

If $a \neq 0$, then, from (1.9), P_n separates all ξ_r, for the denominator takes n different values, which are n different points on the ellipse with center 0 going through $\sinh a$ and $i \cosh a$. On the other hand, if $a = 0$, P_n cannot identify two ξ_r's, for, in the case $a = 0$, (1.9) becomes

$$P_n(\xi_r) = \frac{\sin nb}{\sin\left(b + \frac{2r\pi}{n}\right)},$$

and P_n identifying two ξ_r's, say ξ_h and ξ_k (with $h \neq k$), would imply that

$$\sin\left(b + \frac{2\pi k}{n}\right) = \sin\left(b + \frac{2\pi h}{n}\right)$$

(because $\sin nb \neq 0$ by Corollary 2 to Proposition 3), which would imply in turn that b is a multiple of π/n, contradicting the hypothesis. Q.E.D.

Proposition 5:

(a) The set of n^{th} roots of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is made of all diagonal matrices $\begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix}$, where δ is an n^{th} root of 1, and of all matrices

$$\begin{pmatrix} \cos \frac{k\pi}{n} + T & Y \\ Z & \cos \frac{k\pi}{n} - T \end{pmatrix} \tag{1.10}$$

where Y, Z, T, and k satisfy the following constraints:

(C1): T is any complex number and $YZ = -\left(T^2 + \sin^2 \frac{k\pi}{n}\right)$
[This means exactly that the determinant of (1.10) is 1];

(C2): k is even and $1 \le k \le n - 1$.

(b) The set of n^{th} roots of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathbf{Sl}_2(\mathbf{C})$ is made of all diagonal matrices $\begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix}$, where δ is an n^{th} root of -1, and of all matrices of the form (1.10) satisfying constraint (C1) above, constraint (C2) being replaced by constraint (C3):

(C3): k is odd and $1 \le k \le n - 1$.

Proof:

(a) Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be an n^{th} root of the identity in $\mathbf{Sl}_2(\mathbf{C})$, and let $Q = \alpha + \delta$. By (1.3), we have

$$\begin{pmatrix} \alpha P_n(Q) - P_{n-1}(Q) & \beta P_n(Q) \\ \gamma P_n(Q) & \delta P_n(Q) - P_{n-1}(Q) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.11}$$

If $P_r(Q) = 0$, then $\beta = \gamma = 0$ and $\alpha = 1/\delta$, which implies that α is an n^{th} root of 1, and we have a root which is a diagonal matrix $\begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix}$ as desired. We will therefore suppose that $P_n(Q) = 0$. From Proposition 3, we know that

$$Q = 2 \cos \frac{k\pi}{n} \text{ for some } k \text{ in } \{1, 2, \ldots, n - 1\}.$$

It is clear from (1.11) that β and γ obey no other constraints than $\alpha\delta - \beta\gamma = 1$. On the other hand, α and δ are determined by:

$$\text{(A)} \quad \alpha + \delta = Q; \qquad \text{(B)} \quad P_{n-1}(Q) = -1. \tag{1.12}$$

Using (1.8) to work out the value of $P_{n-1}(Q) = P_{n-1}(2 \cos k\pi/n)$ we obtain

$$P_{n-1}\ 2\left(\cos \frac{k\pi}{n}\right) = \frac{\sinh \frac{(n-1)k\pi i}{n}}{\sinh \frac{k\pi i}{n}} = \frac{\sin \frac{(n-1)k\pi}{n}}{\sin \frac{k\pi}{n}} = (-1)^{k+1}. \tag{1.13}$$

It follows from (1.12)-(B) and (1.13) that k must be even. [Remark: the constraint "k is even and $1 \le k \le n - 1$" in (C2) implies that in (1.10) $n \ge 3$; therefore, $P_2(Q) \neq 0$ and the identity matrix has no square roots of the form (1.10)]. Finally, (1.12)-(A) implies that if $P_n(Q) = 0$, then the diagonal of

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is of the form of the diagonal of (1.10). Moreover, constraint (C1) is satisfied since $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is in $\mathrm{Sl}_2(\mathbf{C})$. As matrices of the form (1.10) are clearly n^{th} roots of the identity matrix (to see this, apply Proposition 1), we have all n^{th} roots of the identity with trace a zero of P_n. This completes the proof of (a).

(b) The proof runs parallel to the proof of (a). The constraint (1.12)-(B) is to be replaced by $P_{n-1}(Q) = 1$, which, by (1.13), implies that k is odd. [Remark: The fact that k is odd allows $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ to have infinitely many roots of any order in $\mathrm{Sl}_2(\mathrm{C})$, as opposed to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which has only two square roots in $\mathrm{Sl}_2(\mathbf{C})$.] Q.E.D.

We now hold all the necessary results to give a complete description of all n^{th} roots of any element of $\mathrm{Sl}_2(\mathbf{C})$.

Theorem A: Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbf{C}),$$

n be any positive integer, $t = (a - d)/2$, and $\chi = a + d$. Then the set of all n^{th} roots of A in $\mathrm{Sl}_2(\mathbf{C})$ is described as follows:

Case 1. $A = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The n^{th} roots of A are exactly the conjugates of $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$, where μ is an n^{th} root of ± 1. [Remark: When A is the identity and $n = 2$,

$$\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is in the center of $\mathrm{Sl}_2(\mathbf{C})$ and thus has no proper conjugates; this is why the identity has only two square roots. Apart from this case A has infinitely many n^{th} roots for each n.]

Case 2. A is not the identity and $\chi = 2$.

There are only one or two root(s), depending on the parity of n; this (these) root(s) is (are)

$$(\sigma/n)\begin{pmatrix} a + (n - 1) & b \\ c & d + (n - 1) \end{pmatrix}, \tag{1.14-A}$$

where σ is ± 1 if n is even and $+1$ if n is odd.

Case 3. $A \neq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ but $\chi = -2$.

There are no roots in $\mathrm{Sl}_2(\mathbf{C})$ if n is even and only one root if n is odd, in which case this root is

$$(1/n)\begin{pmatrix} a - (n - 1) & b \\ c & d - (n - 1) \end{pmatrix}. \tag{1.14-B}$$

Case 4. $\chi \neq \pm 2$.

There are exactly n distinct n^{th} roots. If we set

$$\mu_k = \left(\operatorname{argcosh} \frac{x}{2}\right) + 2k\pi i \quad \text{and} \quad M_k = \frac{\sinh \mu_k/n}{\sinh \mu_k},$$

then these n^{th} roots are $A_0, \ldots, A_{n-1}$, where

$$A_k = \begin{pmatrix} \cosh \frac{\mu_k}{n} + tM_k & bM_k \\ cM_k & \cosh \frac{\mu_k}{n} - tM_k \end{pmatrix}. \tag{1.14-C}$$

Proof of Theorem A:

Throughout, $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ will represent an n^{th} root of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = x + w$.

Case 1. We will consider only the case of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, as the case $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ follows immediately from it. We must prove that the set of conjugates of $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$, where μ is an n^{th} root of 1, is the set of roots described by Proposition 5(a). Let us write $\mathscr{R}$ (for $\mathscr{R}$oot) for the set described by (1.5)-(a) and $\mathscr{C}$ (for $\mathscr{C}$onjugate) for the set of conjugates of $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$.

First, we need a detailed description of $\mathscr{C}$; a direct calculation yields

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \mu\alpha\delta - \frac{\beta\gamma}{\mu} & -2\alpha\beta \sinh(\ln \mu) \\ 2\gamma\delta \sinh(\ln \mu) & -\left(\beta\gamma\mu - \frac{\alpha\delta}{\mu}\right) \end{pmatrix}, \tag{1.15}$$

where $\alpha\delta - \beta\gamma = 1$. If we use the identity

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} \frac{x+w}{2} + \frac{x-w}{2} & y \\ z & \frac{x+w}{2} - \frac{x-w}{2} \end{pmatrix}$$

to rewrite (1.15), we obtain

$$\begin{pmatrix} \cosh(\ln \mu) + \Gamma \sinh(\ln \mu) & -2\alpha\beta \sinh(\ln \mu) \\ 2\gamma\delta \sinh(\ln \mu) & \cosh(\ln \mu) - \Gamma \sinh(\ln \mu) \end{pmatrix} \tag{1.16}$$

where $\Gamma = \alpha\delta + \beta\gamma$. If $\mu = e^{2K\pi i/n}$, $K = 0, \ldots, n - 1$, then (1.15) becomes

$$\begin{pmatrix} \cos \frac{2K\pi}{n} + i\Gamma \sin \frac{2K\pi}{n} & -2\alpha\beta i \sin \frac{2K\pi}{n} \\ 2\gamma\delta i \sin \frac{2K\pi}{n} & \cos \frac{2K\pi}{n} - i\Gamma \sin \frac{2K\pi}{n} \end{pmatrix}. \tag{1.17}$$

Matrix (1.17) characterizes the elements of $\mathscr{C}$ and entails the detailed description of $\mathscr{C}$ that we now use.

We first show that $\mathscr{C} \subset \mathscr{R}$. If $K = 0$, (1.17) is the identity which is trivially in $\mathscr{R}$. If $1 \le 2K \le (n - 1)$, it is trivial to show that (1.17) has the form (1.10) (see Proposition 5) by solving

$$\begin{pmatrix} \cos \frac{2K\pi}{n} + i\Gamma \sin \frac{2K\pi}{n} & -2\alpha\beta i \sin \frac{2K\pi}{n} \\ 2\gamma\delta i \sin \frac{2K\pi}{n} & \cos \frac{2K\pi}{n} - i\Gamma \sin \frac{2K\pi}{n} \end{pmatrix}$$

$$= \begin{pmatrix} \cos \frac{k\pi}{n} + T & Y \\ Z & \cos \frac{k\pi}{n} - T \end{pmatrix} \tag{1.18}$$

with k, T, Y, Z as unknowns. Finally, if $n \le 2K \le 2(n - 1)$, then (1.17) is a matrix with inverse of the form (1.17) for a value of K for which $0 \le 2K \le (n - 1)$; since $\mathscr{R}$ is closed for inversion, (1.17) is in $\mathscr{R}$.

<u>We next show that $\mathscr{R} \subset \mathscr{C}$.</u> All matrices $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$, where μ is an n^{th} root of 1, are trivially in $\mathscr{C}$. Let us consider the system (1.18) with left-hand side as unknown (that is, K, α, β, γ, δ are unknown) and Γ set to $\alpha\delta + \beta\gamma$. Let us set $K = k/2$. [Note that the left-hand side of (1.18) is a typical member of $\mathscr{C}$, and that the right-hand side is a typical member of $\mathscr{R}$. Moreover, the left-hand side of (1.18) is the left-hand side of (1.15) rearranged.]

If $\sin(2K\pi/n) = 0$, the left-hand side of (1.18) is $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, which is trivially in $\mathscr{C}$ $\left[\text{note that } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ occurs only if } n \text{ is even and } K = n/2\right]$. Therefore, we will suppose that $\sin(2K\pi/n) \neq 0$. We wish to show that the elements of $\mathscr{R}$ of the form of the right-hand side of (1.10) are in $\mathscr{C}$, that is, that (1.18), with the left-hand side as unknown, has a solution. This is achieved through showing that the following system has a solution, where (b) comes from $i\Gamma = T$ [see (1.18)], and (C) and (D) from the nondiagonal terms of (1.18):

$$\left\{\begin{array}{ll} a) & \alpha\delta - \beta\gamma = 1 \\[2ex] b) & \alpha\delta + \beta\gamma = \dfrac{T}{i \sin \frac{k\pi}{n}} \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{ll} A) & \alpha\delta = \dfrac{T + i \sin \frac{k\pi}{n}}{2i \sin \frac{k\pi}{n}} \\[2ex] B) & \beta\gamma = \dfrac{T - i \sin \frac{k\pi}{n}}{2i \sin \frac{k\pi}{n}} \end{array}\right\}$$

$$\left\{\begin{array}{ll} C) & \alpha\beta = \dfrac{-Y}{2i \sin \frac{k\pi}{n}} \\[2ex] D) & \gamma\delta = \dfrac{Z}{2i \sin \frac{k\pi}{n}} \end{array}\right\}$$

and where

$$YZ + T^2 + \sin^2 \frac{k\pi}{n} = 0 \text{ (Constraint C1, Proposition 5).}$$

The subsystem (A, B, C) has the following solution in terms of α:

$$\left\{\begin{array}{l} \beta = \dfrac{-Y}{2\alpha i \sin \frac{k\pi}{n}} \\[2ex] \gamma = \dfrac{-\alpha\left(T - i \sin \frac{k\pi}{n}\right)}{Y} \\[2ex] \alpha = \dfrac{T + i \sin \frac{k\pi}{n}}{2\alpha i \sin \frac{k\pi}{n}} \end{array}\right\} \tag{1.19}$$

[Note that, if $Y = 0$, we may use (D) to express γ in terms of α, since $Y = Z = 0$ is possibly only when (1.18) is $\begin{pmatrix} \mu & 0 \\ 0 & 1/\mu \end{pmatrix}$, a case which is trivially in $\mathscr{C}$; therefore, we assume that $|Y| + |Z| \neq 0$ and, without loss of generality, that $Y \neq 0$.] Constraint

$$YZ + T^2 + \sin^2 \frac{k\pi}{n} = 0$$

precisely means that the solutions (1.19) are compatible with (D). Case 1 has thus been established.

Case 2. A is not the identity but $\chi = 2$.

Then, by (1.2),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} xP_n(X) - P_{n-1}(X) & yP_n(X) \\ zP_n(X) & wP_n(X) - P_{n-1}(X) \end{pmatrix}. \tag{1.20}$$

The Möbius transformation defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{PSl}_2(\mathbf{C})$ ("The projective special linear group of degree 2 over $\mathbf{C}$") has a unique fixed point as $\chi = 2$ (see [5]); therefore, the one defined by $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$, the n^{th} iteration of which is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, has also a unique fixed point; thus, we have $X = \pm 2$. On the other hand,

$$\chi_n(-2) = -2 \text{ if } n \text{ is odd,}$$

as is easily checked. From $\chi_n(X) = \chi = 2$, we see that in the case where n is odd we must have $X = 2$. Therefore, from (1.20) and

$$P_n(\pm 2) = n(\pm 1)^{n+1} = \begin{cases} n & \text{if } n \text{ is odd} \\ \pm n & \text{if } n \text{ is even} \end{cases}$$

we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \begin{pmatrix} nx - (n-1) & ny \\ nz & nw - (n-1) \end{pmatrix} & \text{if } n \text{ if odd,} \\ \begin{pmatrix} \pm nx - (n-1) & \pm ny \\ \pm nz & \pm nw - (n-1) \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

Solving then for x, y, z, w in terms of a, b, c, d yields

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{cases} (1/n)\begin{pmatrix} a + (n-1) & b \\ c & d + (n-1) \end{pmatrix} & \text{if } n \text{ is odd,} \\ (\pm 1/n)\begin{pmatrix} a + (n-1) & b \\ c & d + (n-1) \end{pmatrix} & \text{if } n \text{ is even,} \end{cases}$$

which is exactly (1.14-A).

Case 3. $A \neq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ but $\chi - 2$.

As in Case 2, we must have $X = \pm 2$; however, since

$$\chi_n(\pm 2) = 2(\pm 1)^{n+1} = \chi,$$

we must have $X = -2$ and n odd. Moreover, we then have

$$P_n(X) = n \quad \text{and} \quad P_{n-1}(X) = -(n-1).$$

The result follows immediately from (1.20).

Case 4. $\chi \neq \pm 2$.

X is a zero of $\chi_n - \chi$, say (see Proposition 2),

$$X = \xi_k = 2 \cosh \frac{\operatorname{argcosh}(\chi/2) + 2k\pi i}{n} = 2 \cos \frac{\arccos(\chi/2) + 2k\pi}{n}; \tag{1.21}$$

consequently, from

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}^n = \begin{pmatrix} xP_n(X) - P_{n-1}(X) & yP_n(X) \\ zP_n(X) & wP_n(X) - P_{n-1}(X) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we obtain the following possibilities for $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$:

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix} = \begin{pmatrix} \dfrac{a + P_{n-1}(\xi_k)}{P_n(\xi\ )} & \dfrac{b}{P_n(\xi_k)} \\ \dfrac{c}{P_n(\xi_k)} & \dfrac{d + P_{n-1}(\xi_k)}{P_n(\xi_k)} \end{pmatrix} \tag{1.22}$$

[(1.22) uses tacitly Corollary 2 of Proposition 3 in using $P\ (\xi\ )$ in the denominator.] We first show that each of the n matrices defined by (1.22), ($k = 0, \ldots, n - 1$), is an n^{th} root of A in $\mathbf{Sl}_2(\mathbf{C})$ (see Lemma 1 below). Then we show that these matrices are all different (see Lemma 4 below, which requires Lemmas 2 and 3).

Lemma 1: $x_k w_k - y_k z_k = 1$ $\left[\text{the possible values of } \begin{pmatrix} x & y \\ z & w \end{pmatrix} \text{ obtained from Proposition 2 are all in } \mathbf{Sl}_2(\mathbf{C})\right]$.

Proof: Let us set

$$u = \frac{\operatorname{argcosh}(\chi/2) + 2k\pi i}{n}.$$

Then

$$\frac{1}{P_n(\xi_k)} = \frac{\sinh u}{\sinh nu} \text{ by (1.8) and (1.22), and } \frac{\xi_k}{2} = \cosh u \text{ by (1.22),}$$

which gives

$$\begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix} = \begin{pmatrix} \cosh u + t\dfrac{\sinh u}{\sinh nu} & b\dfrac{\sinh u}{\sinh nu} \\ c\dfrac{\sinh u}{\sinh nu} & \cosh u - t\dfrac{\sinh u}{\sinh nu} \end{pmatrix}.$$

Therefore,

$$x_k w_k - y_k z_k = \cosh^2 u - (bc + t^2)\frac{\sinh^2 u}{\sinh^2 nu},$$

but

$$bc + t^2 = bc + \frac{\chi^2 - 4ad}{4} = \left(\frac{\chi}{2}\right)^2 - 1 = \cosh^2 nu - 1 = \sinh^2 nu,$$

whence the result. This completes the proof of Lemma 1.

Lemma 2: $x_k + y_k = \xi_k$.

Proof: From (1.22), we have

$$x_k + y_k = \frac{\chi + 2P_{n-1}(\xi_k)}{P_n(\xi_k)} \tag{1.23}$$

But (see the Remark following Proposition 2),

$$\chi = 2T_n\left(\cosh\left(\frac{\operatorname{argcosh}(\chi/2) + 2k\pi i}{n}\right)\right) = 2T_n\left(\frac{\xi_k}{2}\right);$$

therefore, from (1.6), we have

$$\chi = P_{n+1}(\xi_k) - P_{n-1}(\xi_k),$$

which, by definition (1.1), gives

$$\chi = \xi_k P_n(\xi_k) - 2P_{n-1}(\xi_k).$$

Substituting this value of X into (1.23) yields the result and completes the proof of Lemma 2.

Lemma 3: $\begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix} = \begin{pmatrix} \xi_k/2 + t/P_n(\xi_k) & b/P_n(\xi_k) \\ c/P_n(\xi_k) & \xi_k/2 - t/P_n(\xi_k) \end{pmatrix}$ $\left(\text{Recall: } t = \dfrac{a - d}{2}\right)$.

Proof: From (1.22) and Lemma 2, we have the linear system

$$\begin{cases} x_k + w_k = \xi_k \\ x_k - w_k = \dfrac{2t}{P_n(\xi_k)}, \end{cases}$$

the solution of which is the required result; thus, Lemma 3 is proved.

Lemma 4: The matrices $\begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix}$ $(k = 0, \ldots, n - 1)$ are all different.

Proof: This is simply a consequence of Lemma 3 and Proposition 4, since P_n separates the ξ_k's. This completes the proof of Lemma 4, and Theorem 4 has thus been proved. Q.E.D.

Remark: The denominator of M_k [see (1.14-C)], which is sinh μ_k, with

$$\mu_k = \text{argcosh}\,\frac{\chi}{2} + 2k\pi i,$$

does not depend on k because, if we set $s = \chi/2$, we have

$$\sinh \mu_k = \pm\sqrt{s^2 - 1},$$

where the sign is chosen so as to agree with the principal value of argcosh s; note that M_k does not depend on the choice of this principal value.

In the same fashion, we have

$$\sinh \frac{\mu_k}{n} = \pm\sqrt{s_k^2 - 1}$$

for the numerator of M_k when we set $s_k = \cosh(\mu_k/n)$. Thus, we have

$$M_k = \pm\sqrt{s_k^2 - 1}/\sqrt{s^2 - 1},$$

and, clearly, only the numerator of this expression depends on k.

Roots in $\mathsf{Gl}_2(\mathbf{C})$

Let us conclude with the computation of roots in $\mathsf{Gl}_2(\mathbf{C})$. For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{Gl}_2(\mathbf{C}),$$

let δ be one of the two square roots of det A; we will write δ_+ for δ, δ_- for $-\delta$, A_+ for A/δ_+ and A_- for A/δ_-. Clearly, A_+ and A_- are in $\mathsf{Sl}_2(\mathbf{C})$.

We first observe that the n^{th} roots of A in $\mathsf{Gl}_2(\mathbf{C})$ are elements of ΦB with:

$$\left\{\begin{matrix} \Phi \text{ an } n^{\text{th}} \text{ root of } \delta_+ \text{ and } B \text{ an } n^{\text{th}} \text{ root of } A_+ \\ \text{or} \\ \Phi \text{ an } n^{\text{th}} \text{ root of } \delta_- \text{ and } B \text{ an } n^{\text{th}} \text{ root of } A_- \end{matrix}\right\}. \tag{2.1}$$

It is clear that an element ΦB is an n^{th} root of A, for

$$(\Phi B)^n = \Phi^n B^n = \left\{\begin{matrix} \delta_+ A_+ \\ \text{or} \\ \delta_- A_- \end{matrix}\right\} = A.$$

Conversely, all n^{th} roots of A are of this form, for let $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be an n^{th} root of A and τ be one of the two square roots of $(xw - yz)$; then

$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}^n = \tau^n \begin{pmatrix} x/\tau & y/\tau \\ z/\tau & w/\tau \end{pmatrix}^n$$

from which we get

$$(1/\tau)^n A = \begin{pmatrix} x/\tau & y/\tau \\ z/\tau & w/\tau \end{pmatrix}^n.$$

The determinant of the right-hand side being

$$\left(\frac{xw - yz}{\tau^2}\right)^n = 1,$$

we have that $\tau^n = \delta_\pm$; thus,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \tau \begin{pmatrix} x/\tau & y/\tau \\ z/\tau & w/\tau \end{pmatrix}$$

is of the form ΦB.

To obtain all n^{th} roots of A, we shall compute all products ΦB with Φ and B satisfying (2.1); note that, since A_+ and A_- are in $\mathrm{Sl}_2(\mathbf{C})$, Theorem A gives all possible B's. Let us agree that δ is one of the square roots of $(ad - bc)$ for which $(\Re \operatorname{Tr} A_+) \geq 0$.

We first suppose that A *is not a multiple of the identity*. We consider separately three cases:

Case A. $\operatorname{Tr} A_+ = 2$ and n is even (say $n = 2k$). By Case 2 of Theorem A, A_+ has two roots in $\mathrm{Sl}_2(\mathbf{C})$ which are of opposite signs [see (1.14-A)]; on the other hand, the roots Φ of δ_+ come in pairs with opposite signs and there are $2k$ of them. If $\Phi_1, \ldots, \Phi_k, -\Phi_1, \ldots, -\Phi_k$ are the n possible values for Φ and A_0 and $-A_0$ are two roots of A_+, then the set

$$\{\Phi_1, \ldots, \Phi_k, -\Phi_1, \ldots, -\Phi_k\}\{A_0, -A_0\} \tag{2.2}$$

contains n elements.

On the other hand, A_- has no n^{th} root (see Case 3 of Theorem A); thus, in this case the products of the form

$$(\text{a root of } \delta_-)(\text{a root of } A_-) \tag{2.3}$$

contribute nothing. A_- has therefore altogether n distinct n^{th} roots and these are the elements of the set (2.2).

Case B. $\operatorname{Tr} A_+ = 2$ and n is odd.

Each of A_+ and A_- has exactly one n^{th} root in $\mathrm{Sl}_2(\mathbf{C})$ (Cases 2 and 3 of Theorem A), namely:

$$\text{The root of } A_+: \quad A_0 = \frac{1}{n}\begin{pmatrix} a/\delta_+ + (n-1) & b/\delta_+ \\ c/\delta_+ & d/\delta_+ + (n-1) \end{pmatrix}$$

$$\text{The root of } A_-: \quad -A_0 = \frac{1}{n}\begin{pmatrix} a/\delta_- - (n-1) & b/\delta_- \\ c/\delta_- & d/\delta_- - (n-1) \end{pmatrix}$$

(since $\delta_+ = -\delta_-$, these two roots are of opposite signs). If $r = |\delta|$ and θ is the argument of δ_+, then the n^{th} roots of δ_+ and δ_- are

$$\text{for } \delta_+: \quad re^{i\theta/n}\{\sigma_0, \ldots, \sigma_{n-1}\},$$

$$\text{for } \delta_-: \quad re^{i(\theta+\pi)/n}\{\sigma_0, \ldots, \sigma_{n-1}\},$$

where $\sigma_0, \ldots, \sigma_{-1}$ are the n n^{th} roots of 1. Note that the second set is the first set multiplied by -1. Therefore, the n^{th} roots of A form the union of the following two sets:

$$X_1 = re^{i\theta/n}\{\sigma_0, \ldots, \sigma_{n-1}\}A_0$$
$$X_2 = re^{i\theta/n}\{-\sigma_0, \ldots, -\sigma_{n-1}\}(-A_0)$$

Clearly, $X_1 = X_2$ and their union contains exactly n elements.

Case C. Tr $A_+ \neq 2$.

Let $\sigma_0, \ldots, \sigma_{-1}$ be the n n^{th} roots of 1, and let B be one of the n n^{th} roots of A_+ [see Case 4 of Theorem A, (1.14)-C)]. Then $\sigma_0 B, \ldots, \sigma_{n-1}B$ are all distinct and each of them is an n^{th} root of A_+ since $(\sigma_k B)^n = B^n = A_+$. It follows from Theorem A, Case 4, that $\sigma_0 B, \ldots, \sigma_n B$ are the n roots of A_+, and therefore that the set of elements of the form

$$(\text{a root of } \delta_+)(\text{a root of } A_+) \tag{2.4}$$

is, using the notation of Case B,

$$re^{i\theta/n}\{\sigma_0, \ldots, \sigma_{n-1}\}\{\sigma_0 B, \ldots, \sigma_{n-1}B\},$$

which is the set

$$re^{i\theta/n}\{\sigma_0 B, \ldots, \sigma_{n-1}B\}; \tag{2.5}$$

this set contains n elements.

If σ is any n^{th} root of -1, a similar argument yields

$$re^{i\theta/n}\{\sigma\sigma_0, \ldots, \sigma\sigma_{n-1}\}\{\sigma\sigma_0 B, \ldots, \sigma\sigma_{n-1}B\}$$

for the set of elements of the form (2.3). This is

$$re^{i\theta/n}\sigma^2\{\sigma_0, \ldots, \sigma_{n-1}\}\{\sigma_0 B, \ldots, \sigma_{n-1}B\},$$

which contains exactly n distinct elements. Now, since σ is an n^{th} root of -1, σ^2 is an n^{th} root of 1; then σ^2 is one of $\sigma_0, \ldots, \sigma_{n-1}$, which implies that the set of elements of the form (2.3) is described by (2.5), which is already the set of elements of the form (2.4). Therefore, A has exactly n distinct n^{th} roots in $\mathrm{Gl}_2(\mathbf{C})$.

The case when A *is a (nonzero) multiple of the identity* is immediate; A has infinitely many n^{th} roots, for if $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, then $A = -a\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has infinitely many n^{th} roots for each n (see Theorem A, Case 1). Hence, we have proved the following theorem, which is our conclusion.

Theorem B: Let A be in $\mathrm{Gl}_2(\mathbf{C})$.

a) If A is a nonzero multiple of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then A has infinitely many n^{th} roots;

b) If A is not a multiple of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then A has exactly n distinct n^{th} roots. They are of the form ΦB satisfying (2.1).

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Appendix

Polynomials P_n and χ_n for $2 \le n \le 20$

n	P_n χ_n
2	x $x^2 - 2$
3	$x^2 - 1$ $x^3 - 3x$
4	$x^3 - 2x$ $x^4 - 4x^2 + 2$
5	$x^4 - 3x^2 + 1$ $x^5 - 5x^3 + 5x$
6	$x^5 - 4x^3 + 3x$ $x^6 - 6x^4 + 9x^2 - 2$
7	$x^6 - 5x^4 + 6x^2 - 1$ $x^7 - 7x^5 + 14x^3 - 7x$
8	$x^7 - 6x^5 + 10x^3 - 4x$ $x^8 - 8x^6 + 20x^4 - 16x^2 + 2$
9	$x^8 - 7x^6 + 15x^4 - 10x^2 + 1$ $x^9 - 9x^7 + 27x^5 - 30x^3 + 9x$
10	$x^9 - 8x^7 + 21x^5 - 20x^3 + 5x$ $x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2 - 2$
11	$x^{10} - 9x^8 + 28x^6 - 35x^4 + 15x^2 - 1$ $x^{11} - 11x^9 + 44x^7 - 77x^5 + 55x^3 - 11x$
12	$x^{11} - 10x^9 + 36x^7 - 56x^5 + 35x^3 - 6x$ $x^{12} - 12x^{10} + 54x^8 - 112x^6 + 105x^4 - 36x^2 + 2$
13	$x^{12} - 11x^{10} + 45x^8 - 84x^6 + 70x^4 - 21x^2 + 1$ $x^{13} - 13x^{11} + 65x^9 - 156x^7 + 182x^5 - 91x^3 + 13x$
14	$x^{13} - 12x^{11} + 55x^9 - 120x^7 + 126x^5 - 56x^3 + 7x$ $x^{14} - 14x^{12} + 77x^{10} - 210x^8 + 294x^6 - 196x^4 + 49x^2 - 2$
15	$x^{14} - 13x^{12} + 66x^{10} - 165x^8 + 210x^6 - 126x^4 + 28x^2 - 1$ $x^{15} - 15x^{13} + 90x^{11} - 275x^9 + 450x^7 - 378x^5 + 140x^3 - 15x$
16	$x^{15} - 14x^{13} + 78x^{11} - 220x^9 + 330x^7 - 252x^5 + 84x^3 - 8x$ $x^{16} - 16x^{14} + 104x^{12} - 352x^{10} + 660x^8 - 672x^6 + 336x^4 - 64x^2 + 2$
17	$x^{16} - 15x^{14} + 91x^{12} - 286x^{10} + 495x^8 - 462x^6 + 210x^4 - 36x^2 + 1$ $x^{17} - 17x^{15} + 119x^{13} - 442x^{11} + 935x^9 - 1122x^7 + 714x^5 - 204x^3 + 17x$
18	$x^{17} - 16x^{15} + 105x^{13} - 364x^{11} + 715x^9 - 792x^7 + 462x^5 - 120x^3 + 9x$ $x^{18} - 18x^{16} + 135x^{14} - 546x^{12} + 1287x^{10} - 1782x^8 + 1386x^6 - 540x^4 + 81x^2 - 2$
19	$x^{18} - 17x^{16} + 120x^{14} - 455x^{12} + 1001x^{10} - 1287x^8 - 924x^6 - 330x^4 + 45x^2 - 1$ $x^{19} - 19x^{17} + 152x^{15} - 665x^{13} + 1729x^{11} - 2717x^9 + 2508x^7 - 1254x^5 + 285x^3 - 19x$
20	$x^{19} - 18x^{17} + 136x^{15} - 560x^{13} + 1365x^{11} - 2002x^9 + 1716x^7 - 792x^5 + 165x^3 - 10x$ $x^{20} - 20x^{18} + 170x^{16} - 800x^{14} + 2275x^{12} - 4004x^{10} + 4290x^8 - 2640x^6 + 825x^4 - 100x^2 + 2$

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