DERIVATION OF A FORMULA FOR $\sum r^k x^r$

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1. Introduction

Many different approaches have been proposed to evaluate the sums of powers of consecutive integers,

$$\sum_{r=0}^{n} r^k.$$

Interest in these sums is very old: the Greeks, the Hindus, and the Arabs had rules for the first few cases. Modern interest in these sums goes back more than 350 years to Faulhaber's (1631) "Academia algebrae." Fermat (1636), Pascal (1654), Bernoulli (1713), Jacobi (1834), and many others have also considered this question. Recent contributions are due to Sullivan [1], Edwards [2], Scott [3], and Khan [4]. Sullivan uses a simple and elegant recursion formula to study this problem. Edwards and Scott make use of a matrix formulation which is very intimately connected to Pascal's triangle and the binomial theorem. Khan introduces a simple integral approach that can be presented in all generality with just a basic knowledge of calculus. The interested reader will find a textbook account in Jordan [5], for example.

The purpose of the present note is to study sums of the type

$$\sum_{r=0}^{n} r^k x^r,$$

where n, $k \geq 0$ are integers and x is an arbitrary parameter (real or complex). The sums of powers of consecutive integers can be obtained from our results, as a special case, by letting $x \to 1$. But since the latter sums ($x = 1$) have been studied extensively in the literature, the main emphasis of the present note will be on the former sums ($x \neq 1$).

2. A Method for Evaluating $\sum r^k x^r$

In this section, we present a calculus-based method for evaluating $\sum r^k x^r$. To our knowledge, this approach has not been discussed before. An alternative approach is to use Sullivan's technique [1] by setting $a_r = x^r$, instead of $a_r = 1$, in his expressions. However, after examination, it was found that this approach is not analytically as transparent as the present approach; thus, the details are not reported here.

Let $x \neq 1$ be an arbitrary real or complex parameter, and note the following identity,

$$\sum_{r=0}^{n} x^r = (1 - x^{n+1})/(1 - x). \tag{1}$$

By k successive applications of the differential operator $D = xd/dx$ to both sides of (1), we immediately obtain

$$\sum_{r=0}^{n} r^k x^r = D^k(1 - x^{n+1})/(1 - x). \tag{2}$$

For $k = 0$, (2) is to give back (1) and so we adopt the convention that $r^0 = 1$ for all r, including the case $r = 0$. The above formula provides a compact analytic expression for the desired sums.

By observing that k applications of D on the right-hand side of (2) produces a result with a common denominator of $(1- x)^{k+1}$, we define a set of polynomials of degree $n + k + 1$, $Q_{n+1}(x; k)$; thus:

$$\sum_{0}^{n} r^k x^r = Q_{n+1}(x; k)/(1 - x)^{k+1}, \tag{3}$$

with

$$Q_{n+1}(x; 0) \equiv 1 - x^{n+1}, \tag{4}$$

from (2) and (1). From this point on, the summation index will be r, unless otherwise specified. A recursion formula in k is obtained by noting that

$$\sum_{0}^{n} r^{k+1} x^r = D \sum_{0}^{n} r^k x^r. \tag{5}$$

Identifying each side of (5) with a Q-polynomial as given in (3) we get

$$Q_{n+1}(x; k + 1) = x[(1 - x)Q'_{n+1}(x; k) + (k + 1)Q_{n+1}(x; k)] \tag{6}$$

for k integer ≥ 1; $Q_{n+1}(x; 0)$ is defined by (4), and a prime denotes differentiation with respect to x.

The first few Q-polynomials are:

$$\begin{aligned} Q_{n+1}(x; 1) &= x - (n + 1)x^{n+1} + nx^{n+2}; \\ Q_{n+1}(x; 2) &= x + x^2 - (n + 1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}; \\ Q_{n+1}(x; 3) &= x + 4x^2 + x^3 - (n + 1)^3 x^{n+1} + (3n^3 + 6n^2 - 4)x^{n+2} \\ &\quad - (3n^3 + 3n^2 - 3n + 1)x^{n+3} + n^3 x^{n+4}. \end{aligned} \tag{7}$$

3. General Properties of $\sum r^k x^r$

An inspection of (7) suggests that the Q-polynomials may be written as x^n times a polynomial of degree k in n, plus a term which is n-independent. Consequently, this property also holds for $\sum r^k x^r$, by (3). To see this more clearly, rewrite (2) as follows:

$$\sum_{0}^{n} r^k x^r = D^k \frac{x^{n+1}}{x - 1} - D^k \frac{1}{x - 1}. \tag{8}$$

The first term on the right-hand side generates x^n times a polynomial of degree k in n and the second term generates a term which is independent of n. As a result, in an effort to display the n-dependence of the right-hand side as explicitly as possible, we rewrite (8) in the form

$$\sum_{0}^{n} r^k x^r = x^n P_k(x; n) + R_k(x), \tag{9}$$

where

$$P_k(x; n) = \sum_{r=0}^{k} a_r^{(k)}(x) n^r \tag{10}$$

is a polynomial of degree k in n, with coefficients $a_r^{(k)}$ which depend on x. The term R is independent of n and so, by setting $n = 0$ in (9), we find that $R_k = -a_0^{(k)}$, except when $k = 0$. Indeed, because of our earlier convention that r^0 be equal to 1 for all $r \geq 0$, the case $k = 0$ has to be handled differently. From (2), with $k = 0$, we find that

$$R_0(x) = -1/(x - 1) \quad \text{and} \quad a_0^{(0)}(x) = x/(x - 1).$$

Finally, with this restriction in mind, we rewrite (9) in the form

$$\sum_0^n r^k x^r = x^n \sum_{r=0}^k a_r^{(k)}(x) n^r - a_0^{(k)}(x) \tag{11}$$

and establish rules to obtain the coefficients $a_r^{(k)}$. To obtain these coefficients, we will use two different methods: A) a method of recursion on k; and B) a method of recursion on n.

A) k-Recursive Method: This method consists in assuming that the $a_r^{(k)}$'s are known for some k. Then, by using (5), the next set of coefficients, $a_r^{(k+1)}$, is determined. By (5) and (9)-(11), we get

$$x^n P_{k+1}(x;\ n) - a_0^{(k+1)} = D[x^n P_k(x;\ n) - a_0^{(k)}]. \tag{12}$$

To reduce this expression, perform the derivative and get

$$x^n[P_{k+1}(x;\ n) - nP_k(x;\ n) - DP_k(x;\ n)] = a_0^{(k+1)}(x) - Da_0^{(k)}(x). \tag{13}$$

The right-hand side of (13) is independent of n but the left-hand side has a factor which grows exponentially with n. Consequently, for (13) to hold for all values of n, with x fixed but arbitrary, we must have

$$a_0^{(k+1)} = Da_0^{(k)}, \tag{14}$$

$$P_{k+1} = nP_k + DP_k. \tag{15}$$

To reduce (15) further, define

$$a_{k+1}^{(k)} \equiv 0, \quad a_{-1}^{(k)} \equiv 0, \tag{16}$$

and use (10) to get

$$\sum_0^{k+1} [a_r^{(k+1)}(x) - a_{r-1}^{(k)}(x) - Da_r^{(k)}(x)] n^r = 0. \tag{17}$$

In order for this expression to hold for all n, with x fixed but arbitrary, we must have

$$a_r^{(k+1)} = a_{r-1}^{(k)} + Da_r^{(k)}. \tag{18}$$

Because of (16), the case $r = 0$ is consistent with (14) above; similarly, for $r = k + 1$, we get

$$a_{k+1}^{(k+1)} = a_k^{(k)}, \tag{19}$$

and so we conclude, from (3), that

$$a_k^{(k)}(x) = x/(x - 1) \tag{20}$$

for all k, including $k = 0$. One significant drawback of this k-recursive approach is that all previous sums must be known in order to determine the k^{th} one. Fortunately, however, using method B, it is possible to determine the k^{th} sum independently from the others.

B) Induction on n: By induction on n, (10) and (11) give

$$\sum_{0}^{n+1} r^k x^r = x^{n+1}P_k(x; n+1) - a_0^{(k)}(x) \tag{21}$$

or, equivalently,

$$(n+1)^k x^{n+1} = x^{n+1}P_k(x; n+1) - x^n P_k(x; n). \tag{22}$$

With (10), this gives

$$x(n+1)^k = x\sum_{0}^{k} a_r^{(k)}(n+1)^r - \sum_{0}^{k} a_r^{(k)} n^r. \tag{23}$$

To simplify the notation in what follows, we will write a_r for $a_r^{(k)}$ because the upper index k is kept fixed.

Using the binomial expansion, (23) becomes

$$x\sum_{j=0}^{k} a_j \sum_{r=0}^{j}\binom{j}{r} n^r - \sum_{j=0}^{k} a_j n^j = x\sum_{j=0}^{k}\binom{k}{j} n^j. \tag{24}$$

For this equation to hold for all n, we must have equality of the coefficients of like powers of n on both sides; hence,

$$a_k = x/(x-1), \tag{25}$$

as observed previously, and

$$a_r = \frac{x}{x-1}\left[\binom{k}{r} - \sum_{j=r+1}^{k}\binom{j}{r} a_j\right], \tag{26}$$

for $0 \le r \le k-1$. We give here the first few a_r's, for arbitrary k; a_k is given by (25), and

$$\begin{aligned} a_{k-1} &= -kx/(x-1)^2, \\ a_{k-2} &= k(k-1)x(x+1)/2(x-1)^3, \\ a_{k-3} &= -k(k-1)(k-2)x(x^2+4x+1)/6(x-1)^4. \end{aligned} \tag{27}$$

Others are determined readily using (26).

To conclude this section, we extend (2) to negative values of n. To do so, first note that the right-hand side of (2) is well defined for all values of n, with k an integer ≥ 0. For $n = -1$, the right-hand member of (2) is zero, so we adopt the convention that

$$\sum_{0}^{-1} r^k x^r = 0 \text{ for all } x \ne 1 \text{ or } 0.$$

For n an integer ≥ 2, we let

$$\begin{aligned} \sum_{0}^{-n} r^k x^r &\equiv D^k(x^{-n+1} - 1)/(x-1) = -D^k\left(\frac{1}{x^{n-1}} - 1\right)\Big/\left(x\left(\frac{1}{x} - 1\right)\right) \\ &= -D^k \sum_{1}^{n-1} x^{-r}, \end{aligned} \tag{28}$$

i.e.,

$$\sum_{0}^{-n} r^k x^r \equiv -\sum_{1}^{n-1}(-r)^k x^{-r}, \tag{29}$$

with $\sum_1^0 \equiv 0$ on the right-hand side.

Now set $n = -1$ in (11) to obtain

$$x^{-1}\sum_{0}^{k} a_r^{(k)}\cdot(-1)^r - a_0^{(k)} = 0, \tag{30}$$

i.e.,

$$(x - 1)a_0^{(k)} = \sum_{1}^{k}(-1)^r a_r^{(k)} \text{ for all } x \neq 1. \tag{31}$$

This interesting property can be observed in the special cases that follow.

4. Interesting Special Cases

In this section, results for $k = 1, 2, 3, 4$, and 5 are presented.

To begin with, we let $x = 2$ and find the following sums:

$$\begin{aligned}
&\sum_{0}^{n} r\cdot 2^r = 2[2^n(n-1)+1];\\
&\sum_{0}^{n} r^2\cdot 2^r = 2[2^n(n^2-2n+3)-3];\\
&\sum_{0}^{n} r^3\cdot 2^r = 2[2^n(n^3-3n^2+9n-13)+13];\\
&\sum_{0}^{n} r^4\cdot 2^r = 2[2^n(n^4-4n^3+18n^2-52n-75)+75];\\
&\sum_{0}^{n} r^5\cdot 2^r = 2[2^n(n^5-5n^4+30n^3-130n^2+375n-541)+541].
\end{aligned} \tag{32}$$

There is an interesting regularity in the coefficients of n in the parentheses; for example, the absolute value of the coefficient of n^0 is equal to the sum of the absolute values of the coefficients of all the higher-order terms.

The second sum in (32) belongs to a class of sums where the summand $r^k x^r$ is symmetric under the interchange of r and k: $r^k\cdot k^r$. Such sums have an intrinsic appeal and we give a few examples below:

$$\begin{aligned}
&\sum_{0}^{n} r^2\cdot 2^r = \frac{2}{1^2}[2^n(n^2-2n+3)-3];\\
&\sum_{0}^{n} r^3\cdot 3^r = \frac{3}{2^3}[3^n(4n^3-6n^2+12n-11)+11];\\
&\sum_{0}^{n} r^4\cdot 4^r = \frac{4}{3^4}[4^n(27n^4-36n^3+90n^2-132n+95)-95].
\end{aligned} \tag{33}$$

The case $\sum_0^n r^1\cdot 1^r$ has to be handled differently because (1) does not hold for $x = 1$; we shall discuss this type of situation in C) below.

Other interesting results are now given in A)-C).

A) For $x = -1$:

$$\sum_{0}^{n}(-1)^r r = \frac{1}{4}[(-1)^n(2n+1)-1];$$

$$\sum_0^n (-1)^r r^2 = \frac{1}{2}[(-1)^n(n^2 + n)] = (-1)^n \sum_0^n r; \tag{34}$$

$$\sum_0^n (-1)^r r^3 = \frac{1}{8}[(-1)^n(4n^3 + 6n^2 - 1) + 1].$$

B) For $x = i$ imaginary, we get, for example,

$$\sum_0^n i^r r^2 = \frac{1}{2}[i^n(n^2 + 2n + i(1 - n^2)) - i]. \tag{35}$$

If the real and imaginary terms are gathered separately, for n even, two identities are obtained. The identity for the real terms gives back the second equation of (34) and that for the imaginary terms gives the new identity,

$$\sum_0^{n/2-1} (-1)^r (2r + 1)^2 = [(-1)^{n/2}(1 - n^2) - 1]. \tag{36}$$

C) For $x = 1$: In order to obtain the sums of powers of consecutive integers, take the limit $x \to 1$ in (3) and get

$$\sum_0^n r^k = \lim_{x \to 1} \frac{Q_{k+1}(x;\ k)}{(1 - x)^{k+1}} = \frac{(-1)^{k+1}}{(k + 1)!} \lim_{x \to 1} \frac{d^{k+1}}{dx^{k+1}} Q_{n+1}(x;\ k) \tag{37}$$

after $k + 1$ applications of l'Hôpital's rule. For $k = 0, 1, 2, 3$, equations (7) give, respectively:

$$\begin{aligned}
&\lim_{x \to 1} \frac{d}{dx} Q_{n+1}(x;\ 0) = -(n + 1);\\
&\lim_{x \to 1} \frac{d^2}{dx^2} Q_{n+1}(x;\ 1) = n(n + 1);\\
&\lim_{x \to 1} \frac{d^3}{dx^3} Q_{n+1}(x;\ 2) = -n(n + 1)(2n + 1);\\
&\lim_{x \to 1} \frac{d^4}{dx^4} Q_{n+1}(x;\ 3) = 6n^2(n + 1)^2.
\end{aligned} \tag{38}$$

Insertion of these results in (37) gives the expected results for the appropriate sums. The present technique is, however, somewhat cumbersome to handle. Indeed, k derivatives are first required to find $Q_{n+1}(x;\ k)$ followed by $k + 1$ additional ones in order to compute the limit. Cases with $x = 1$ can be handled easily with Khan's technique or by the method of induction on n presented earlier. Indeed, by observing, from (8), that

$$\lim_{x \to 1} D^k(x^{n+1} - 1)/(x - 1)$$

is a polynomial of degree $k + 1$ in n, we may write

$$\sum_0^n r^k = \sum_0^{k+1} a_r^{(k)} n^r - a_0 \tag{39}$$

and proceed as before.

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