INTEGRAL TRIANGLES AND CIRCLES

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Having noticed that Pythagorean triangles have integral diameter circumcircles and integral diameter incircles, A. Hunter was prompted to inquire as to whether there were any integer-sided, non-right-angled triangles having integral diameter incircles or integral diameter circumcircles. After a couple of weeks of Diophantine analysis, the answer to these questions was found to be in the affirmative in both cases.

The first solutions found in this way were:

SIDES			DIAMETER
Incircle Problem			
7	15	20	4
Circumcircle Problem			
182	560	630	650

On investigating the circumcircle problem, M. Kovarik devised the construction in Figure 1 which shows how that problem may be solved by means of Pythagorean triangles.

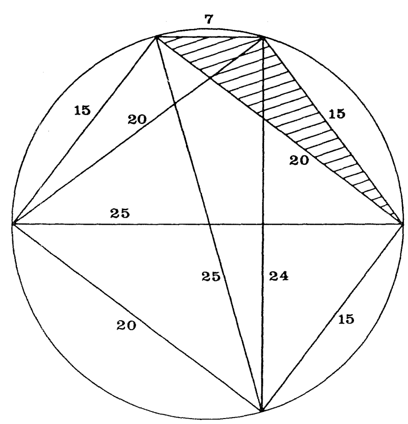

FIGURE 1

Kovarik's construction for the integer-sided scalene triangle 7,15,20 having an integral diameter circumcircle

The first integer-sided scalane triangle produced by Kovarik's method was the 7,15,20 triangle, which happened to coincide with the first solution to the integral diameter incircle problem found above. This prompted Hunter to inquire whether other triangles constructed by Kovarik's method had integral diameter incircles.

Generalizations of Kovarik's construction are shown in Figures 2-5. The two Pythagorean triplets a,b,c and r,s,t are scaled to have the common hypotenuse, ct.

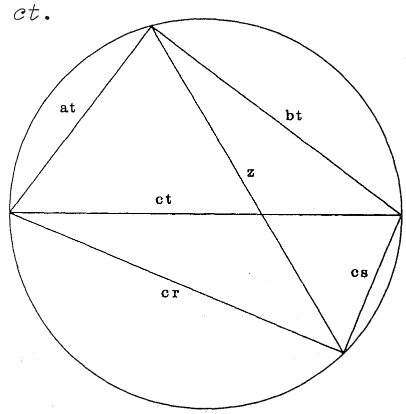

FIGURE 2

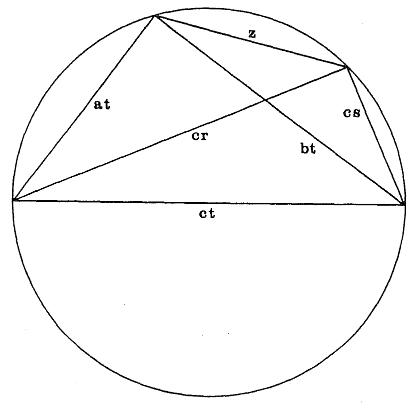

FIGURE 3

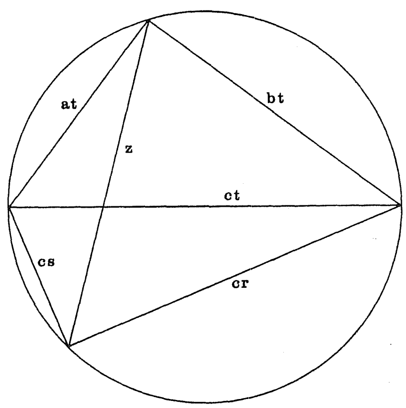

FIGURE 4

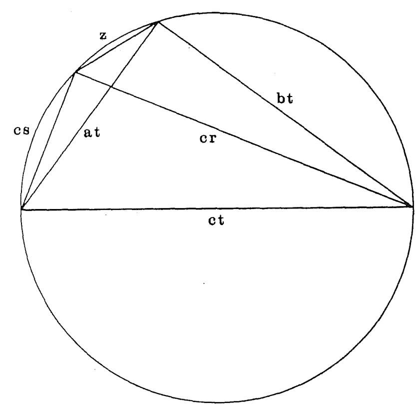

FIGURE 5

FIGURES 2-5

Four constructions for integer-sided, non-right-angled triangles having an integral diameter circumcircle, based on the pair of Pythagorean triplets a,b,c and r,s,t

For a cyclic quadrilateral, Ptolemy's theorem states that the sum of the products of the opposite sides is equal to the product of the diagonals. It follows that the values of z corresponding to Figures 2, 3, 4, and 5 are

$$as + br, \quad |as - br|, \quad ar + bs, \text{ and } |ar - bs|,$$

respectively, and are clearly integers. There are two non-right-angled, integer-sided triangles x,y,z for each of Figures 2-5 as given in Table 1.

TABLE 1

Triangle No.	x	y	z
1	at	cr	$as + br$
2	bt	cs	$as + br$
3	at	cr	$\|as - br\|$
4	bt	cs	$\|as - br\|$
5	at	cs	$ar + bs$
6	bt	cr	$ar + bs$
7	at	cs	$\|ar - bs\|$
8	bt	cr	$\|ar - bs\|$

It remained to investigate the diameters of the incircles of these triangles.

The diameter of the incircle of a triangle whose sides are x,y,z is given by

$$d = \sqrt{\frac{(x + y - z)(y + z - x)(z + x - y)}{x + y + z}}$$

Substitution of the values of x, y, and z given in Table 1 yields the incircle diameters given in Table 2.

TABLE 2

Triangle No.	Incircle Diameter
1	$ar - (c - b)(t - s)$
2	$bs - (c - a)(t - r)$
3	$ar - (c + b)(t - s)$
4	$bs - (c - a)(t + r)$
5	$as - (c - b)(t - r)$
6	$br - (c - a)(t - s)$
7	$as - (c - b)(t + r)$
8	$br - (c + a)(t - s)$

Clearly, the diameters of all incircles are integers. The integral triangles and circles (itacs) generated from the Pythagorean triplets

$$\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

are shown in Figure 6. When the two Pythagorean triplets are equal, triangles numbered 1 and 2 become isosceles, numbers 3 and 4 diminish to a point, 5 and 6 become right-angled, and 7 and 8 are scalene. The itacs generated from

$$\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 17 \end{bmatrix}$$

are shown in Figure 7.

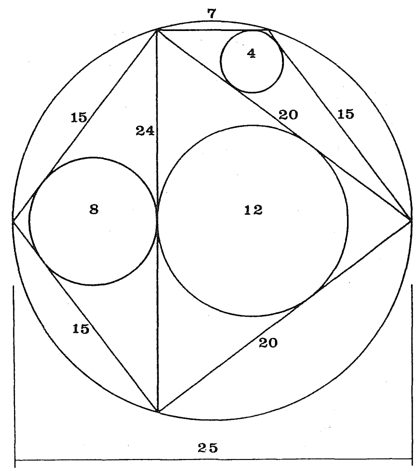

FIGURE 6

Itacs based on a,b,c equal to r,s,t equal to 3,4,5

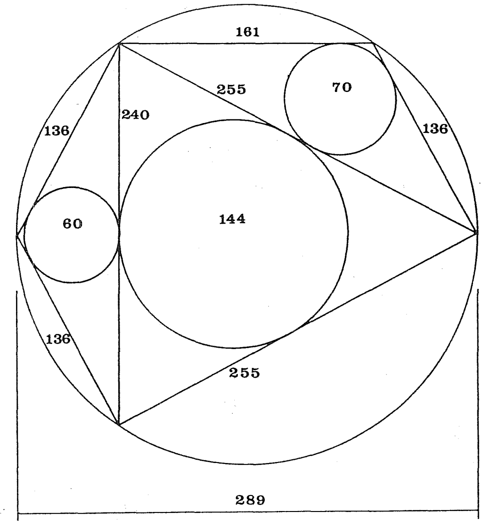

FIGURE 7

Itacs based on a,b,c equal to r,s,t equal to 8,15,17

Where a,b,c and r,s,t are independent Pythagorean triplets, all eight triangles are, in general scalene. Itacs generated from

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix}$$

are shown in Figures 8 and 9 and itacs generated from

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 17 \end{bmatrix}$$

are shown in Figures 10 and 11.

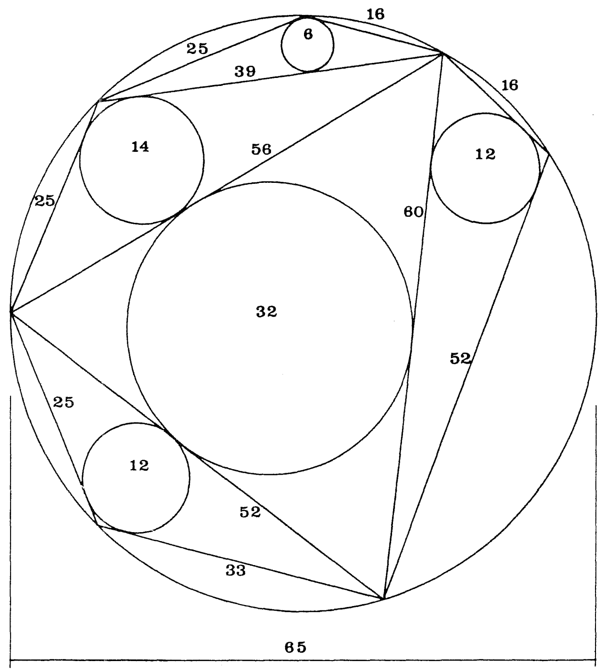

FIGURE 8

Itacs based on a,b,c equal to 3,4,5 and r,s,t equal to 5,12, 13 (5 incircles)

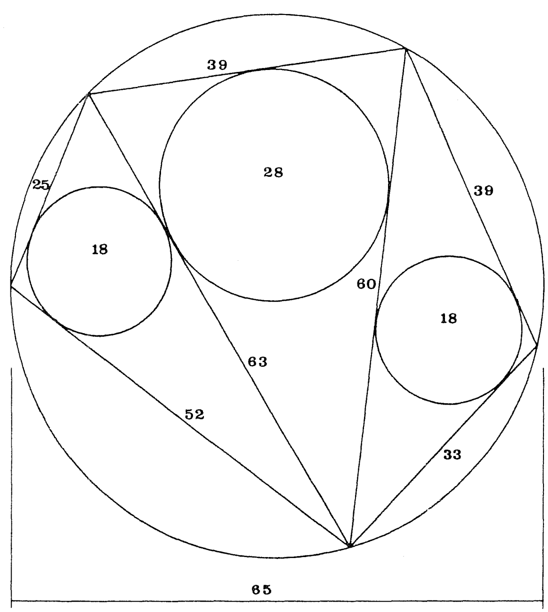

FIGURE 9

Itacs based on a,b,c equal to 3,4,5 and r,s,t equal to 5,12,13 (3 incircles)

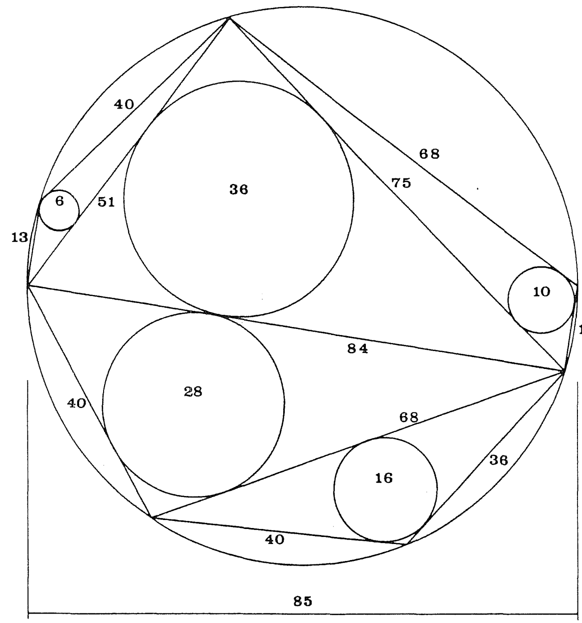

FIGURE 10

Itacs based on a,b,c equal to 3,4,5 and r,s,t equal to 8,15,17 (5 incircles)

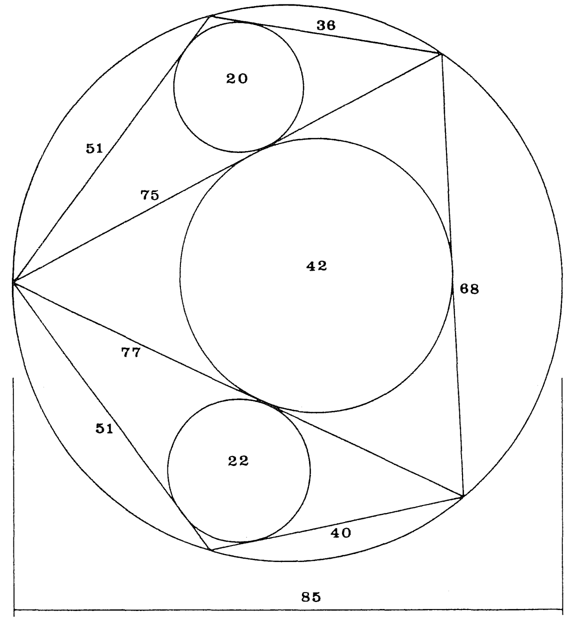

FIGURE 11

Itacs based on a,b,c equal to 3,4,5 and r,s,t equal to 8,15,17 (3 incircles)

Triangles arising in itacs are also Heronian; that is, have integral areas. The area of a triangle is given by

$$A = \frac{xyz}{2d},$$

where d is the diameter of the circumcircle, and x, y, and z are the sides. Substituting from Table 1, we obtain expressions such as

$$A = \frac{ar(as + br)}{2}.$$

For Pythagorean triplets a,b,c and r,s,t, ab and rs are both even, so A is always an integer.

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Announcement

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The FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Wake Forest University, Winston-Salem, N.C., from July 30 to August 3, 1990. This Conference is sponsored jointly by the Fibonacci Association and Wake Forest University.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1990, while manuscripts are due by May 1, 1990. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

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