# A DIOPHANTINE EQUATION WITH GENERALIZATION

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In [1] the authors showed that the diophantine equation $Nb^2 = c^2 + N + 1$ does not admit any integral solution except for the trivial case $N = -1$ and $b = c = 0$. At the end of the proof, a conjecture about its generalization was made, namely that

$$Nb^2 = c^2 + N(4k + 1) + 1 \tag{1}$$

would not yield any nontrivial solutions.

In this paper we give a new proof of the original equation. We also prove that (1) does not have a solution when $N$ is a positive integer. A counter-example is given to show that there may exist infinitely many solutions of the general equation when $N$ takes negative values, so the conjecture in (1) was not correct.

Omitting the trivial case when $b = c = 0$, we show that

$$N = \frac{c^2 + 1}{b^2 - 1}$$

is not an integer for all integral values of $b$ and $c$. Suppose that $N$ is an integer. We consider two cases. Suppose $b$ is even. This means that $b^2 - 1 \equiv 3 \pmod 4$, which implies that there exists at least one prime $p \equiv 3 \pmod 4$ such that $p$ divides $b^2 - 1$. This in turn leads to $c^2 + 1 \equiv 0 \pmod p$, which is impossible since $-1$ is a quadratic nonresidue $(\mathrm{mod}\ p)$. If $b$ is odd, then $b^2 - 1 \equiv 0 \pmod 8$, so $c^2 + 1 = 0 \pmod 8$, which is also impossible.

To show that (1) has solutions when $N$ is negative, take $N = -2$. The equation becomes $8k - 2b^2 = c^2 - 1$, which has infinitely many solutions given by

$$b = 2m, \quad c = 2n + 1, \quad \text{and } k = m^2 + \frac{n(n + 1)}{2},$$

where $m$, $n$ are arbitrary integers. The reader can easily generate infinitely many solutions by selecting other specific negative values of $N$.

*Theorem:* The diophantine equation $Nb^2 = c^2 + N(4k + 1) + 1$ does not admit any solution when $N > 0$.

*Proof:* We consider five cases:

1. Let $N \equiv 3 \pmod 4$. There is a prime factor $p$ of $N$ such that $p \equiv 3 \pmod 4$. This implies $c^2 + 1 \equiv 0 \pmod p$, which is impossible.

2. $N \equiv 1 \pmod 4$. Let $N = 4t + 1$ with $t \geq 0$. The equation becomes

$$(4t + 1)b^2 = c^2 + 4M + 2,$$

where $M = 4tk + t + k$. This equation is solvable only if the congruence $b^2 - c^2 \equiv 2 \pmod 4$ is solvable. But since $b^2 - c^2 \equiv 0, 1, 3 \pmod 4$ for all possible choices of $b$ and $c$, $b^2 - c^2 \equiv 2 \pmod 4$ is not solvable.

3. $N \equiv 0 \pmod 4$. This implies $c^2 + 1 \equiv 0 \pmod 4$, which is impossible.

4. $N \equiv 2 \pmod 8$. Let $N = 8t + 2$ with $t \geq 0$. The equation becomes

$$(8t + 2)b^2 = c^2 + (8t + 2)(4k + 1) + 1,$$

which implies $2b^2 - c^2 \equiv 3 \pmod 8$ is solvable. Since $x^2 \equiv 0, 1, 4 \pmod 8$ for all integers $x$, $2b^2 - c^2 \equiv 0, 1, 2, 4, 6, 7 \pmod 8$; thus, $2b^2 - c^2 \equiv 3 \pmod 8$ is not solvable.

5. $N \equiv 6 \pmod 8$. Let $N = 8t + 6 = 2(4t + 3)$. Then $N$ contains a prime factor $p$, where $p \equiv 3 \pmod 4$. Thus, the solution of the equation is not possible for the reason discussed in Case 1 above, and the proof is complete.

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## Reference

1. David A. Anderson & Milton W. Loyer. "The Diophantine Equation $Nb^2 = c^2 + N + 1$." *Fibonacci Quarterly 17.1* (1979):69-70.

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