ON FIBONACCI PRIMITIVE ROOTS

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1. Introduction

In [6] D. Shanks introduced the concept of a Fibonacci Primitive Root (FPR) mod p, i.e., an integer g which is a primitive root mod p and satisfies the congruence $g^2 \equiv g + 1 \bmod p$. He proved some properties of FPR's, for instance: If for a prime p, $p \neq 5$, there is an FPR mod p, then $p \equiv \pm 1 \bmod 10$. He also made the following conjecture:

Let
$$F(x) = \text{card}\{p \le x\colon p \in \mathbb{P},\ \exists_g\ g \text{ is FPR mod } p\},$$
and
$$\pi(x) = \text{card}\{p \le x\colon p \in \mathbb{P}\}.$$

Conjecture: As $x \to \infty$,
$$\frac{F(x)}{\pi(x)} \sim C,$$
where $C = \dfrac{27}{38} \displaystyle\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p(p-1)}\right)$.

Note that
$$\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p(p-1)}\right) = 0.3739558136\ldots$$
is Artin's constant.

By a theorem of DeLeon [3] and deep-lying work of Göttsch [4] using methods of Hooley [5] on Artin's conjecture, we will prove the Conjecture above on the assumption of a certain Riemann hypothesis, namely,

Theorem: Let $\rho = (1 + \sqrt{5})/2$, ζ be a primitive $2n^{\text{th}}$ root of unity, where n is a positive integer, and C be defined as in the Conjecture. If the Riemann hypothesis holds for all fields $\mathbb{Q}(\sqrt[n]{\rho}, \zeta)$, then
$$\frac{F(x)}{\pi(x)} = C + O\left(\frac{\log \log x}{\log x}\right).$$

2. Preliminaries

Let (f_n) be the classical Fibonacci sequence, i.e.,
$$f_0 = 0,\ f_1 = 1,\ f_{n+2} = f_{n+1} + f_n \quad (n \ge 0).$$
An easy pigeon-hole principle argument yields the periodicity of (f_n) mod m for any integer $m > 1$. Let $\lambda(m)$ be the length of the smallest period mod m.

Lemma 1: ([3], Theorem 1) Let $p \neq 5$ be a prime. Then there exists an FPR mod p iff $p \equiv \pm 1 \bmod 10$ and $\lambda(p) = p - 1$.

The following lemma has been proved by Göttsch [4]. A rather obvious generalization which, however, is more accessible has been given by Antoniadis [1].

Lemma 2: ([4], Kor. 2.10; [1], Satz 2 and Kor. 4) Let

$$A(x) = \text{card}\{p \le x\colon p \equiv \pm 1 \bmod 10,\ \lambda(p) = p - 1\}.$$

Under the assumption made in the Theorem, we have

$$A(x) = C\frac{x}{\log x} + O\left(\frac{x \log\log x}{(\log x)^2}\right),$$

where C is defined in the Conjecture.

It should be remarked that, without assuming the Riemann hypothesis, the applied methods only give upper bounds for $A(x)$ (see [4]). These are useless with regard to the Conjecture.

3. Proof of the Theorem

Since there is an FPR mod 5, we have, by Lemma 1, for $x \ge 5$,

$$F(x) = 1 + A(x).$$

Applying Lemma 2, we get

$$F(x) = C\frac{x}{\log x} + O\left(\frac{x \log\log x}{(\log x)^2}\right).$$

By the Prime Number Theorem (see, e.g., [2]),

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Thus,

$$F(x) = C\pi(x) + O\left(\frac{x \log\log x}{(\log x)^2}\right),$$

which implies the Theorem.

References

1. J. A. Antoniadis. "Über die Periodenlänge mod p einer Klasse rekursiver Folgen." *Arch. Math.* *42* (1984):242-252.
2. H. Davenport. *Multiplicative Number Theory*. 2nd ed. New York-Heidelberg-Berlin: Springer-Verlag, 1980.
3. M. J. DeLeon. "Fibonacci Primitive Roots and the Period of the Fibonacci Numbers modulo p." *Fibonacci Quarterly* *15* (1977):353-355.
4. G. Göttsch. "Über die mittlere Periodenlänge der Fibonacci-Folgen modulo p." Dissertation, Hannover, 1982.
5. Ch. Hooley. "On Artin's Conjecture." *J. Reine Angew. Math.* *225* (1967):209-220.
6. D. Shanks. "Fibonacci Primitive Roots." *Fibonacci Quarterly* *10* (1972):163-168.

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