A PROOF FROM GRAPH THEORY FOR A FIBONACCI IDENTITY

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One of the beauties of mathematics is its consistency. To find, serendipitously, a verification of a result from an area other than the one being studied is an unexpected bonus. One such bonus is a proof of the Fibonacci identity

(1) $$f_{n+2} = f_{i+1}f_{n-i} + f_{i+2}f_{n-i+1}, \quad 1 \le i \le n - 2$$

which arose during a count of maximal independent sets in trees.

First, we need some definitions from graph theory [1].

A graph G is a nonempty finite set of points, or vertices, V, along with a prescribed set E of unordered pairs of distinct points of V, called edges. We write $G = (V, E)$.

If two distinct points, x and y, of a graph are joined by an edge, they are said to be adjacent, and we write x adj y.

A walk of a graph G is a finite sequence of points such that each point of the walk is adjacent to the point of the walk immediately preceding it and to the point immediately following it. If the last vertex of the walk is the same as the first, the walk is closed. If a closed walk contains at least three distinct points and all are distinct except the first and last, then we have a cycle. A graph is acyclic if it contains no cycles. A walk is a path if it contains no cycles. A walk is a path if all the points are distinct.

A graph is connected if every pair of points is joined by a path.

A tree is a connected, acyclic graph.

The degree of a point v in G, denoted deg v, is the number of edges incident with v.

An endpoint or end vertex or leaf of a tree is a vertex of degree one. (Every tree has at least two endpoints.) An interior point of a tree is any vertex with a degree greater than one.

An independent set for graph G is a set of vertices with the property that no two vertices in the set are adjacent.

A maximal independent set (MIS) of G is an independent set which is contained in no other independent set of G.

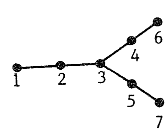

FIGURE 1

For the tree in Figure 1, {1, 3, 6} is an independent set; {1, 3, 6, 7} and {2, 4, 5} are maximal independent sets. Note that not all maximal independent sets are the same size. Also note that any vertex v is either included in a given maximal independent set or adjacent to a vertex in that maximal independent set.

It was hoped that the number and sizes of maximal independent sets would be a key to the structure of a tree. Although that was not the case, it was while counting the maximal independent sets of a narrow class of trees that the counterexample was found, along with the Fibonacci identity (1).

Let T be a tree with n vertices. Let $p(T)$ be the tree obtained by adding one edge and one end vertex to each vertex of T. Then $p(T)$ has $2n$ vertices, and is called the expanded tree of T, and T is the reduced or core tree of $p(T)$. The expanded tree has exactly n end vertices. If T is a tree with $2n$ vertices and exactly n end vertices, then each of the end vertices (with its adjoining edge) can be removed to obtain the core tree, which we call $p^{-1}(T)$. Figure 2 shows a core tree with its expanded tree. The added vertices are circled.

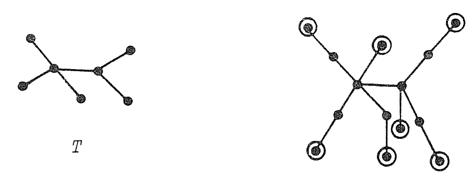

FIGURE 2

If e is an endpoint of tree T which is adjacent to a vertex x of degree 2, call e a remote end vertex.

Now consider only the set of trees that are expanded trees of n-paths, n = 1, 2, 3, ... Let us count the number of maximal independent sets for each of these trees.

Let M_T = the number of maximal independent sets of T.

Let T be the expanded tree of an n-path. For each vertex v in T, define $\lambda(v)$ to be the number of maximal independent sets of T that contain v. If v is an interior point (i.e., not an endpoint) and if w is the endpoint adjacent to v, then

$$\lambda(v) + \lambda(w) = M_T,$$

since every maximal independent set must contain either v or w. In particular, if e is a remote end vertex and e adj x, then

$$\lambda(e) + \lambda(x) = M_T.$$

If x adj y, $y \neq e$, then

$$\lambda(y) + \lambda(x) = \lambda(e),$$

since e belongs to every MIS containing y, and if a MIS S contains x, then $(S - \{x\}) \cup \{e\}$ is also a MIS of the proper size, and these are the only MIS's that could possibly contain e.

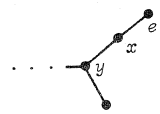

FIGURE 3

By combining these two equalities, we find that $M_T = \lambda(y) + 2\lambda(x)$.

The following proposition states some more facts about relationships among λ-numbers.

Proposition 1: Let T be an expanded tree that looks like the tree in figure 4; that is, e is a remote end vertex, e adj x, x adj y, $y \neq e$, z_1 is the end vertex adjacent to y, z_2 adj y, $z_2 \neq z_1$, $z_2 \neq x$, and z_3 is the end vertex adjacent to z_2. The structure of the rest of T (which is attached at z_2) does not matter.

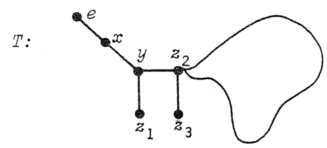

FIGURE 4

Then:

(i) $\lambda(z_3) = 3\lambda(y)$, so that $\lambda(z_3)$ is divisible by 3;

(ii) $\lambda(z_1) = 2\lambda(y) + \lambda(z_2)$;

(iii) $\lambda(z_2)$ is even;

(iv) $\lambda(z_1)$ is even;

(v) $\lambda(y)$ and M_T have the same parity;

(vi) $\lambda(v)$ is independent of the number of remote end vertices attached to v for any $v \in T$ that is an interior point.

In addition,

(vii) $\lambda(e) = M_{T-\{e,x\}}$.

Proof:

(i) For every MIS S containing y, $z_3 \in S$, $S - \{y\} \cup \{z_1\}$,

$S - \{y, e\} \cup \{z_1, x\}$,

so $\lambda(z_3) = 3\lambda(y)$, and 3 divides $\lambda(z_3)$.

(ii) This can be proved in two ways:

(a) $M_T = \lambda(z_2) + \lambda(z_3) = \lambda(z_2) + 3\lambda(y)$ by (i);

$M_T = \lambda(y) + \lambda(z_1)$.

The difference of these two equations is

$$0 = 2\lambda(y) + \lambda(z_2) - \lambda(z_1)$$

or

$$\lambda(z_1) = 2\lambda(y) + \lambda(z_2).$$

(b) For every MIS S containing y,

$$z_1 \in S - \{y, e\} \cup \{x, z_1\}$$

and

$$z_1 \in S - \{y\} \cup \{z_1\}.$$

z_1 is also in every MIS containing z_2.

(iii) Let T_1 be the part of T containing e, x, y, z_1, z_2, z_3, and

$$T_2 = (T - T_1) \cup \{z_2\}.$$

$\lambda(z_2)$ = (number of MIS's in T_1 containing z_2) × (number of MIS's in T_2 containing z_2) = 2 × (number of MIS's in T_2 containing z_2). The two sets in T_1 are $\{z_1, z_2, e\}$ and $\{z_1, z_2, x\}$. Thus, 2 divides $\lambda(z_2)$.

(iv) follows immediately from (ii) and (iii); and (v) follows from (iv) and

$$M_T = \lambda(y) + \lambda(z_1).$$

(vi) Let $e_1, e_2, \ldots, e_k$ be the remote end vertices attached to some interior point v in T, with e_1 adj x_1, x_i adj v, $i = 1, 2, \ldots, k$. Then every MIS containing v must also contain all the e_i's, and if a MIS contains even one x_i, then v is *not* a member of that set. Thus, $\lambda(v)$ is not affected by the size of k.

(vii) Let $\lambda'(v)$ be the number of MIS's containing v in $T - \{e, x\}$, for any $v \in T - \{e, x\}$.

$$\lambda(e) + \lambda(x) = M_T$$

and

$$\lambda(x) + \lambda(y) = \lambda(e)$$

so that

$$2\lambda(e) = M_T + \lambda(y)$$

so

$$\lambda(e) = \frac{M_T + \lambda(y)}{2}; \qquad (2)$$

also

$$M_T = 2M_{T-\{e,x\}} - \lambda'(y)$$

since for every MIS S in $T - \{e, x\}$ we have, in T, the MIS $S \cup \{e\}$ and the MIS $S \cup \{x\}$, except when S contains y.

But (vi) implies that $\lambda(y) = \lambda'(y)$, so

$$\lambda(y) = \lambda'(y) = 2M_{T-\{e,x\}} - M_T. \qquad (3)$$

(2) and (3) lead to

$$\lambda(e) = \frac{M_T + 2M_{T-\{e,x\}} - M_T}{2} = M_{T-\{e,x\}}.$$

Now to determine M_T: The expanded trees of n-paths have exactly two remote end vertices.

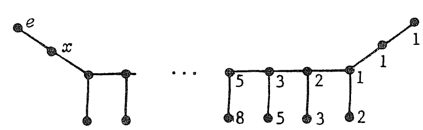

FIGURE 5

We will call the central $n - 2$ points of the core tree the central path, and will find the λ-numbers for all points of the central path, as well as for the nonremote end vertices.

Starting at the right-hand end of the central path, we label each vertex of the central path and each corresponding end vertex with the number of maximal independent sets containing the point that include only points that have been previously labeled, or points only "to the right" of the given point. Points "to the right" of an end vertex shall include the point in the core tree to which it is connected. These labels will be elements of the Fibonacci sequence (1, 1, 2, 3, 5, 8, ... is the Fibonacci sequence, where the n^{th} term, f_n, equals the sum of the two previous terms: $f_n = f_{n-1} + f_{n-2}$).

Since a vertex in the central path is contained in exactly the same number of maximal independent sets to its right as the most recently labeled end vertex, its λ-number will be the same as the label of that end vertex. (Note that because of the order of labeling described above, these two points are *not* adjacent.)

Since any end vertex can be added to a MIS containing the most recently labeled end vertex (to the right) or to a MIS containing the vertex in the central path adjacent to the most recently labeled end vertex, the label of the end vertex in question will be the sum of the labels of those two previously labeled vertices.

Then since the farthest remote end vertex to the right and the point adjacent to it can each only be in one MIS to the right, we see (refer to Fig. 5) that the labels of the interior points are indeed a Fibonacci sequence. The labels of the end vertices form a Fibonacci sequence but start with the second term.

A portion of the tree in Figure 5 with labels that will correspond to the following discussion is shown in Figure 6.

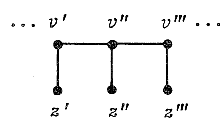

FIGURE 6

Define $r(w)$ to be the number of MIS's containing vertex w and points "to the right" of w.

For a vertex v' in the central path, v' is in the same number of maximal independent sets to its right as z'' is in, where z'' is the endpoint adjacent to v'', v'' adj v' and v'' to the right of v'. $r(z') = r(v'') + r(z'')$ since, if z' is in a MIS S (containing points only "to the right" of z'), then either $z'' \in S$ or $v'' \in S$.

Therefore,

$$r(v') = r(z'') = r(z''') + r(v''') = r(v'') + r(v''').$$

Thus, if we number the vertices of the central path from left to right by $v_{n-2}, v_{n-3}, \ldots, v_2, v_1$, and the end vertices by $z_{n-2}, z_{n-3}, \ldots, z_2, z_1$, where z_i adj v_i, then

$$r(v_i) = f_{i+1} \quad \text{and} \quad r(z_i) = f_{i+2}.$$

Then,

$$\lambda(v_{n-2}) = f_{n-1} \quad \text{and} \quad \lambda(z_{n-2}) = 2f_n$$

(since for every MIS "to the right" we could add either the remote end vertex e or the adjacent point x) and hence,

$$M_T = \lambda(v_{n-2}) + \lambda(z_{n-2}) = f_{n-1} + 2f_n$$
$$= f_{n-1} + f_n + f_n = f_{n+1} + f_n = f_{n+2}.$$

Now label each point v_i and end point z_i in a similar manner from the left by $\ell(v_i)$ and $\ell(z_i)$.

$$\lambda(v_i) = r(v_i) \cdot \ell(v_i) \quad \text{and} \quad \lambda(z_i) = r(z_i) \cdot \ell(z_i).$$

Note that $\ell(v_i) = f_{n-i}$ since v_i is the $n - i - 1^{\text{st}}$ point from the left, and

$$\ell(z_i) = f_{n-i+1}.$$

Since

$$\lambda(v_i) + \lambda(z_i) = M_T, \; 1 \le i \le n - 2,$$

we have the following well-known [2] number theoretic result.

Theorem 3.24: $f_{n+2} = f_{i+1}f_{n-i} + f_{i+2}f_{n-i+1}$ for $1 \le i \le n - 2$.

For more general expanded trees, we follow a similar procedure. If v is a member of the core tree and deg $v = k$, then $\lambda(v)$ is the product of $k - 1$ labels—one from each of the $k - 1$ branches incident with v. If v adj z, z an end vertex, then $\lambda(z)$ is also the product of $k - 1$ labels. In this general case, the labels will not always be elements of the Fibonacci sequence, but each individual label will be obtained as the sum of the two previous labels in the same branch. It is not necessary to find all labels for every point in order to find M_T. Only the λ-numbers for one end vertex and its adjacent point are needed.

As an example, in Figure 7 is a tree with 20 vertices. The points e and y are the ones for which the λ-numbers are being found. We are labeling from the endpoints of the separate branches *toward* the vertices e and y, in the order of the indices on the v's.

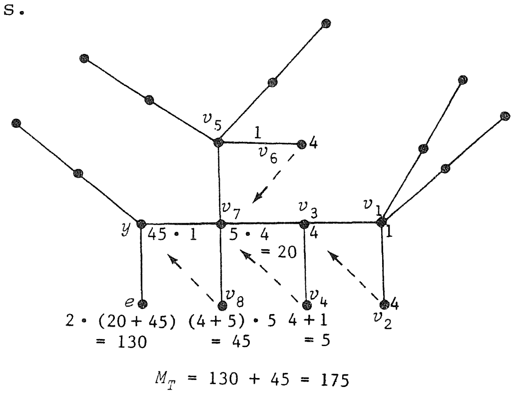

FIGURE 7

v_1 is in only one MIS to the right.

v_2 is in 4 MIS's to the right—there are two choices on each of the paths leading to the remote end vertices for 2 • 2 MIS's.

v_3 is in the same MIS's to the right as v_2.

v_4 is in the same MIS's as v_1 or the same MIS's as v_2, for a total of 1 + 4, or 5.

v_5 and v_6 are like v_1 and v_2, respectively, when labeling from the end of their branch, i.e., "from above."

v_7 gets a label of 4 from above (the same as v_6) and a label of 5 from the right (the same as v_4) for a total of $5 \cdot 4$ MIS's.

v_8 is in 4 + 5 sets to the right (the sum of labels from v_3 and v_4) and in 1 + 4 sets from above (the sum of labels from v_5 and v_6) for a total of $9 \cdot 5$, or 45.

$\lambda(y) = 1 \cdot 45$, the product of the number of MIS's to the left and the number of MIS's to the right (from the v_8 label).

$\lambda(e) = 2 \cdot (20 + 45)$, with 2 being the number of MIS's to the left and 20 + 45 being the sum of the labels from v_7 and v_8.

$M_T = \lambda(y) + \lambda(e) = 45 + 130 = 175.$

If we look at the triangular array of λ-numbers for the central $n - 2$ vertices of the core tree of the expanded tree of an n-path, we see a triangle whose entries grow along diagonals in a Fibonacci-like manner. Figure 8 shows the first 3 trees and gives the λ-numbers of the circled vertices. In the following chart, $2n$ is the number of vertices of the expanded n-path.

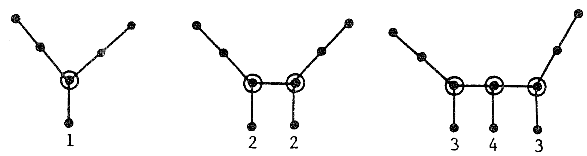

FIGURE 8

	$2n$
1	6
2 2	8
3 4 3	10
5 6 6 5	12
8 10 9 10 8	14
13 [16] 15 15 16 13	16
12 26 24 25 24 26 21	18
16 = 2 × 8 ⋮	⋮

Notice that the triangle is symmetric about a vertical line through its center. The two outer diagonals are the Fibonacci sequence without the first term. All other diagonals are Fibonacci-like in that each term, starting with the third is the sum of the two terms immediately preceding it in the diagonal. Also, each diagonal is a set of multiples of the first element, and the members correspond to multiples of the shortened Fibonacci sequence seen in the outer diagonals.

Also notice that if there are $2n$ vertices in the tree, there are n vertices in its core tree and $n - 2$ vertices of that core tree will not be adjacent to a remote end vertex in the original tree. Therefore, there will be $n - 2$ vertices to label and $n - 2$ numbers in the row of the triangle associated with $2n$ (see Fig. 8).

Now, the remarkable coincidences of the triangle can be understood if we recall the way in which the vertices of the central path are labeled. Each label is the product of the number of MIS's to the right and the number of MIS's to the left. However, the numbers of MIS's to the right for the central path are just the Fibonacci numbers, starting with the second term. Likewise, because of symmetry, the numbers of MIS's to the left are also the Fibonacci numbers, starting with the second term. So for the $n - 2$ elements of the triangle row associated with $2n$, we have $f_{n-1-i} \cdot f_i$, $2 \le i \le n - 1$.

For example, if we let the factor on the left represent the number of MIS's to the left and the factor on the right represent the number of MIS's to the right, we see that the row associated with $2n = 14$ and $n - 2 = 5$ is:

$$1 \cdot 8 \qquad 2 \cdot 5 \qquad 3 \cdot 3 \qquad 5 \cdot 2 \qquad 8 \cdot 1$$

Certainly, the growth of the numbers related to maximal independent sets in this special class of trees is related to Fibonacci numbers and patterns, and the study of one enhances the other.

References

1. F. Harary. *Graph Theory*. Reading: Addison-Wesley, 1969.
2. I. Niven & H. S. Zuckerman. *An Introduction to the Theory of Numbers*. New York: Wiley & Sons, 1960, pp. 98-99.

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