ADVANCED PROBLEMS AND SOLUTIONS

Edited by

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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-437 *Proposed by L. Kuipers, Sierre, Switzerland*

Let x, y, n be Natural numbers, where n is odd. If

$$L_n/L_{n+2} < x/y < L_{n+1}/L_{n+3}, \text{ show that } y > 1/5L_{n+4}. \qquad (*)$$

Are there fractions, x/y, satisfying (*) for which $y < L_{n+4}$?

H-438 *Proposed by H.-J. Seiffert, Berlin, Germany*

Define the Fibonacci polynomials by

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \text{ for } n \geq 2.$$

Show that, for all odd integers $n \geq 3$,

$$\int_{-\infty}^{+\infty} \frac{dx}{F_n(x)} = \frac{\pi}{n}\left(1 + 1/\cos\frac{\pi}{n}\right).$$

H-439 *Proposed by Richard Andre-Jeannin, ENIS BP.W, Tunisia*

Let p be a prime number ($p \neq 2$) and m a Natural number. Show that

$$L_{2p^m} + L_{4p^m} + \cdots + L_{(p-1)p^m} \equiv 0 \pmod{p^{m+1}}.$$

SOLUTIONS

Some Difference

H-414 *Proposed by Larry Taylor, Rego Park, New York (Vol. 25, no. 3, August, 1987)*

Let j, k, m, and n be integers. Prove that

$$F_{m+j}F_{n+k} = F_{m+k}F_{n+j} - F_{k-j}F_{n-m}(-1)^{m+j}.$$

Solution by Tad White, UCLA, Los Angeles, CA

The proof is by induction on each of the four variables. Let $P(j, k, m, n)$ denote the above equality. It is trivial to verify this equality for $j, k, m, n \in \{0, 1\}$. We thus need only show that

(i) $P(j - 2, k, m, n)$ and $P(j - 1, k, m, n) \Rightarrow P(j, k, m, n)$, and

(ii) $P(j + 2, k, m, n)$ and $P(j + 1, k, m, n) \Rightarrow P(j, k, m, n)$,

and similarly for the other three variables. The proofs are essentially identical for each variable, so we will present only the induction on j here for brevity.

Notice that the equality $P(j, k, m, n)$ can be written in determinant form:

$$\begin{vmatrix} F_{m+k} & F_{m+j} \\ F_{n+k} & F_{n+j} \end{vmatrix} = F_{k-j}F_{n-m}(-1)^{m+j}.$$

Using the Fibonacci recursion relation, the determinant on the left can be rewritten as

$$\begin{vmatrix} F_{m+k} & F_{m+j-1} + F_{m+j-2} \\ F_{n+k} & F_{n+j-1} + F_{n+j-2} \end{vmatrix}.$$

By linearity of the determinant in the second column, this is

$$\begin{vmatrix} F_{m+k} & F_{m+j-1} \\ F_{n+k} & F_{n+j-1} \end{vmatrix} + \begin{vmatrix} F_{m+k} & F_{m+j-2} \\ F_{n+k} & F_{n+j-2} \end{vmatrix}$$

which, by the induction hypothesis, equals

$$\begin{aligned} &F_{k-j+1}F_{n-m}(-1)^{m+j-1} + F_{k-j+2}F_{n-m}(-1)^{m+j-2} \\ &= (F_{k-j+2} - F_{k-j+1})F_{n-m}(-1)^{m+j} \\ &= F_{k-j}F_{n-m}(-1)^{m+j}, \end{aligned}$$

as required. The induction in the negative direction is the same, except that one uses the Fibonacci recursion relation in the reverse direction.

Also solved by P. Bruckman, P. Filipponi, L. Kuipers, J. Mahon, F. Makri & D. Antzoulakos, B. Prielipp, H.-J. Seiffert, and the proposer.

A Little Reciprocity

H-416 *Proposed by Gregory Wulczyn, Bucknell U. (retired), Lewisburg, PA (Vol. 25, no. 4, November, 1987)*

(1) If $\left(\frac{p}{5}\right) = 1$, show that $\begin{cases} .5(L_{p-1} + F_{p-1}) \equiv 1 \pmod{p}, \\ .5(L_{p+1} - F_{p+1}) \equiv 1 \pmod{p}. \end{cases}$

(2) If $\left(\frac{p}{5}\right) = -1$, show that $\begin{cases} .5(L_{p-1} + F_{p-1}) \equiv -1 \pmod{p}, \\ .5(L_{p+1} - F_{p+1}) \equiv -1 \pmod{p}. \end{cases}$

Solution by Lawrence Somer, Washington, D.C.

It is well known that

$$F_{p-(5/p)} \equiv 0 \pmod{p} \quad \text{and} \quad F_p \equiv (5/p) \pmod{p}.$$

It is also known that

$$L_n = F_{n-1} + F_{n+1}.$$

It follows from the law of quadratic reciprocity that $(p/5) = (5/p)$ if p is a prime greater than 2.

(1) Suppose $(p/5) = 1$. Then $p \neq 2$. It follows that

$$F_{p-(5/p)} = F_{p-1} \equiv 0 \pmod{p}$$

and

$$F_p \equiv (5/p) \equiv 1 \pmod{p}.$$

Then

$$F_{p+1} = F_{p-1} + F_p \equiv 0 + 1 \equiv 1 \pmod{p}.$$

Thus,

$$L_{p-1} = F_{p-2} + F_p = (F_p - F_{p-1}) + F_p \equiv 1 - 0 + 1 \equiv 2 \pmod{p}$$

and

$$L_{p+1} = F_p + F_{p+2} = F_p + (F_p + F_{p+1}) \equiv 1 + 1 + 1 \equiv 3 \pmod{p}.$$

Hence,

$$.5(L_{p-1} + F_{p-1}) \equiv .5(2 + 0) \equiv 1 \pmod{p}$$

and

$$.5(L_{p+1} - F_{p+1}) \equiv .5(3 - 1) \equiv 1 \pmod{p}.$$

(2) Assume that $(p/5) = -1$. First suppose that $p = 2$. Then

$$L_{p-1} = L_1 = 1,\ L_{p+1} = L_3 = 4,\ F_{p-1} = F_1 = 1,\ F_{p+1} = F_3 = 2.$$

Then

$$.5(L_{p-1} + F_{p-1}) = .5(1 + 1) \equiv -1 \pmod{2}$$

and

$$.5(L_{p+1} - F_{p+1}) = .5(4 - 2) \equiv -1 \pmod{2}.$$

Now suppose that $p \neq 2$. It follows that

$$F_{p-(5/p)} = F_{p+1} \equiv 0 \pmod{p}$$

and

$$F_p \equiv (5/p) \equiv -1 \pmod{p}.$$

Then

$$F_{p-1} = F_{p+1} - F_p \equiv 0 - (-1) \equiv 1 \pmod{p}.$$

Hence,

$$L_{p-1} = F_{p-2} + F_p = (F_p - F_{p-1}) + F_p \equiv -1 - 1 - 1 \equiv -3 \pmod{p}$$

and

$$L_{p+1} = F_p + F_{p+2} = F_p + (F_p + F_{p+1}) \equiv -1 + (-1) + 0 \equiv -2 \pmod{p}.$$

Thus,

$$.5(L_{p-1} + F_{p-1}) \equiv .5(-3 + 1) \equiv -1 \pmod{p}$$

and

$$.5(L_{p+1} - F_{p+1}) \equiv .5(-2 - 0) \equiv -1 \pmod{p}.$$

Also solved by P. Bruckman, P. Filipponi, C. Georghiou, L. Kuipers, T. White, and the proposer.

A Mean Problem

H-417 *Proposed by Piero Filipponi, Rome, Italy*
(Vol. 25, no. 4, November, 1987)

Let $G(n, m)$ denote the geometric mean taken over m consecutive Fibonacci numbers of which the smallest is F_n. It can be readily proved that

$$G(n, 2k+1) \qquad (k = 1, 2, \ldots)$$

is not integral and is asymptotic to F_{n+k} (as n tends to infinity).

Show that if n is odd (even), then $G(n, 2k+1)$ is greater (smaller) than F_{n+k}, except for the case $k = 2$, where $G(n, 5) < F_{n+2}$ for every n.

Solution by Paul Bruckman, Edmonds, WA

$$(1) \qquad G(n, 2k+1) = \left(\prod_{j=0}^{2k} F_{n+j}\right)^{\frac{1}{2k+1}}.$$

Hence,

$$\log G(n, 2k+1) = \frac{1}{2k+1}\sum_{j=0}^{2k} \log F_{n+j}$$

$$= \frac{1}{2k+1}\sum_{j=0}^{2k} (\log a^{n+j} - \tfrac{1}{2}\log 5 + \log(1 - (b/a)^{n+j})$$

$$= \frac{1}{2k+1}\sum_{j=0}^{2k} ((n+j)\log a - \tfrac{1}{2}\log 5 + \log(1 - x^{n+j})),$$

where a and b are the usual Fibonacci constants and $x = b/a = -b^2$. (Note that $-1 < x < 0$.) Thus,

$$(2) \qquad \log G(n, 2k+1) = (n+k)\log a - \tfrac{1}{2}\log 5 + \frac{1}{2k+1}\sum_{j=0}^{2k}\log(1 - x^{n+j}).$$

Likewise,

$$(3) \qquad \log F_{n+k} = (n+k)\log a - \tfrac{1}{2}\log 5 + \log(1 - x^{n+k}).$$

We now make the definition:

$$(4) \qquad D(n, k) = \log\left(\frac{G(n, 2k+1)}{F_{n+k}}\right).$$

Thus, it suffices to prove that $D(n, k) > 0$ if n is odd and $D(n, k) < 0$ if n is even, unless $k = 2$, in which case $D(n, 2) < 0$ for all n.

Now, from (2) and (3), we have

$$(5) \qquad D(n, k) = \frac{1}{2k+1}\sum_{j=0}^{2k}\log(1 - x^{n+j}) - \log(1 - x^{n+k}).$$

Expanding into Maclaurin series, we obtain:

$$D(n, k) = \frac{-1}{2k+1}\sum_{j=0}^{2k}\sum_{i=1}^{\infty} i^{-1}(x^{n+j})^i + \sum_{i=1}^{\infty} i^{-1}(x^{n+k})^i$$

$$= \sum_{i=1}^{\infty} i^{-1}x^{ni}\left(x^{ki} - \frac{1}{2k+1}\left(\frac{1 - x^{(2k+1)i}}{1 - x^i}\right)\right),$$

or, after some simplification,

$$D(n,\ k) = \sum_{i=1}^{\infty} i^{-1}x^{(n+k)i}\left(1 - \frac{(-1)^{ki}}{2k+1} \cdot \frac{F_{(2k+1)i}}{F_i}\right). \tag{6}$$

We consider the various possibilities:

Case 1. k is even, $k \ge 2$. Then

$$D(n,\ k) = \sum_{i=1}^{\infty} \frac{x^{(n+k)i}}{i}\left(1 - \frac{F_{(2k+1)i}}{(2k+1)F_i}\right);$$

if, moreover, n is even, $(n + k)$ is even, and the last expression is clearly negative (the first term vanishing if $k = 2$). If n is odd, then

$$D(n,\ k) > b^{2n+2k}\left(\frac{F_{2k+1}}{2k+1} - 1\right) - \tfrac{1}{2}b^{4n+4k}\left(\frac{F_{4k+2}}{2k+1} - 1\right)$$

so

$$\begin{aligned} D(n,\ k) &> b^{2n+2k}\left(\frac{a^{2k+1}}{(2k+1)\sqrt{5}} - 1\right) - \tfrac{1}{2}b^{4n+4k}\left(\frac{a^{4k+2}}{((2k+1)\sqrt{5} - 1)}\right) \\ &\ge \frac{a^{-(2n-1)}}{2(2k+1)\sqrt{5}}(2 - a^{-(2n-1)}) - \tfrac{1}{2}b^{2n+2k}(2 - b^{2n+2k}) \\ &\ge \frac{a^{-(2n-1)}}{2(2k+1)\sqrt{5}}(2 - a^{k-1}) - b^{2n+2k} \ge \frac{a^{-(2n-1)}\ a^{-1}}{2(2k+1)} - b^{2n+2k} \\ &= b^{2n}\left(\frac{1}{4k+2} - b^{2k}\right) > 0 \text{ if } k \ge 4 \text{ (since } a^{2k} > 4k + 2 \text{ if } k \ge 3). \end{aligned}$$

Thus far, we have shown that

(7) $D(n,\ k) < 0$ if k and n are even;

$D(n,\ k) > 0$ if $k \ge 4$ is even, n is odd.

Also, if n is odd,

$$\begin{aligned} D(n,\ 2) &= \sum_{i=2}^{\infty} i^{-1}x^{(n+2)i}\left(1 - \frac{F_{5i}}{5F_i}\right) < -5x^{2n+4} - 20x^{3n+6} \\ &= -5x^{2n+4}(1 + 4x^{n+2}) \le -5x^{2n+4}(1 - 4b^6) < 0. \end{aligned}$$

Thus,

(8) $D(n,\ 2) < 0$ for all n.

Case 2. k is odd, $k \ge 3$. Then

$$D(n,\ k) = \sum_{i=1}^{\infty} i^{-1}x^{(n+k)i}\left(1 - \frac{(-1)^{i}}{2k+1} \cdot \frac{F_{(2k+1)i}}{F_i}\right).$$

If n is odd,

$$D(n,\ k) > b^{2n+2k}\left(1 + \frac{a^{2k+1}}{(2k+1)\sqrt{5}}\right) - \tfrac{1}{2}b^{4(n+k)}\left(\frac{a^{4k+2}}{(2k+1)\sqrt{5}} - 1\right)$$

(continued)

$$\geq \tfrac{1}{2}b^{2(n+k)}(2 + b^{2(n+k)}) + \frac{a^{-(2n-1)}}{2(2k + 1)\sqrt{5}}(2 - a^{-(2n-1)})$$

$$> b^{2(n+k)} + \frac{a^{-(2n-1)}}{2(2k + 1)\ 5}(2 - a^{-1}) = b^{2n}\left(b^{2k} + \frac{1}{4k + 2}\right) > 0.$$

If n is even,

$$D(n, k) = -b^{2(n+k)}\left(1 + \frac{F_{2k+1}}{2k + 1}\right) - \tfrac{1}{2}b^{4(n+k)}\left(\frac{F_{4k+2}}{2k + 1} - 1\right) - \cdots;$$

clearly, $D(n, k) < 0$ in this case. Therefore,

(9) $D(n, k) > 0$ if $k \geq 3$ and n are odd;
$D(n, k) < 0$ if $k \geq 3$ is odd, and n is even.

Combining (7), (8), and (9) yields the desired conclusion:

(10) $D(n, k) < 0$, if n is even, $k \neq 2$;
$D(n, k) > 0$, if n is odd, $k \neq 2$;
$D(n, 2) < 0$ for all n. Q.E.D.

Also solved by the proposer.

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