ON CIRCULAR FIBONACCI BINARY SEQUENCES

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The number of combinations of n elements taken k at a time is given by the binomial coefficient $\binom{n}{k}$. If the n elements are arranged in a circle, any two circular combinations are said to be indistinguishable if one can be obtained by a cyclic rotation of the other. Let $C(n, k)$ denote the number of distinguishable circular combinations of n elements taken k at a time. Using a formula for $C(n, k)$, we consider a problem on circular Fibonacci binary sequences.

We recall that a Fibonacci binary sequence is a $\{0, 1\}$-sequence with no two 1's adjacent. Similarly, a circular Fibonacci sequence is a circular $\{0, 1\}$-sequence with no two 1's adjacent. Let $H(n)$ denote the number of distinguishable circular Fibonacci binary sequences of length n, and let $W(n)$ denote the total number of 1's in all such sequences. The ratio $Q(n) = W(n)/nH(n)$ gives the proportion of 1's in all the distinguishable circular Fibonacci binary sequences of length n. In the case of ordinary Fibonacci binary sequences, this ratio tends to the limit $(5 - \sqrt{5})/10$ as $n \to \infty$ [2]. In the case of circular Fibonacci binary sequences, a similar result can be proved.

For any integer

$$m = p_1^{r_1} p_2^{r_2} \ldots p_j^{r_j} \geq 2,$$

where p_i's are distinct prime numbers and $r_i \geq 1$, let $\phi(m)$ be defined by

$$\phi(m) = m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_j}\right);$$

for $m = 1$, let $\phi(m) = 1$. Thus, ϕ is the Euler totient function. The number $C(n, k)$ of all distinguishable circular combinations of n elements taken k at a time is given by the following formula.

$$C(n, k) = \frac{1}{n} \sum_{1 \leq m | (n, k)} \phi(m)\binom{n/m}{k/m}.$$

(See [1], p. 208.)

Now let $g(n, k)$ denote the number of distinguishable circular Fibonacci binary sequences of length n which contain a total of k 1's. Since each 1 must be followed by a 0 in the sequence,

$$g(n, k) = C(n - k, k).$$

If n is a prime number, the ratio

$$Q(n) = \frac{W(n)}{nH(n)} = \frac{1}{n} \frac{C(n-1, 1) + 2C(n-2, 2) + 3C(n-3, 3) + \cdots}{1 + C(n-1, 1) + C(n-2, 2) + C(n-3, 3) + \cdots}$$

$$= \frac{1 + \binom{n-3}{1} + \binom{n-4}{2} + \cdots}{n\left[1 + 1 + \binom{n-3}{1}/2 + \binom{n-4}{2}/3 + \cdots\right]}$$

Using the following formula (see [3], p. 76),

$$\sum_{k \geq 0} \binom{n-k}{k} x^k = \frac{1}{2^{n+1}s}[(1 + s)^{n+1} - (1 - s)^{n+1}],$$

where $s = \sqrt{1 + 4x}$, one has

$$W(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^{n-1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n-1}\right],$$

$$nH(n) = n - 1 + \left[\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots\right] + \left[\binom{n-2}{0} + \binom{n-3}{1} + \binom{n-4}{2} + \cdots\right]$$

$$= n - 1 + \left(\frac{1 + \sqrt{5}}{2}\right)^{n} + \left(\frac{1 - \sqrt{5}}{2}\right)^{n}.$$

Thus, the limit through prime numbers is

$$\lim_{\substack{n \to \infty \\ n \text{ is prime}}} Q(n) = (5 - \sqrt{5})/10.$$

In general, for any positive integer $n = p_1^{r_1}p_2^{r_2} \ldots p_j^{r_j}$, one has

$$nH(n) = n\left[1 + 1 + \binom{n-3}{1}/2 + \binom{n-4}{2}/3 + \cdots\right]$$

$$+ \sum_{i=1}^{j} \frac{n}{p_i}\,\phi(p_i) \sum_{r \geq 1} \frac{1}{r}\left[\binom{n/p_i - r - 1}{r - 1} + \binom{n/p_i^2 - r - 1}{r - 1} + \cdots + \binom{n/p_i^{r_i} - r - 1}{r - 1}\right]$$

$$+ \sum_{\substack{i,m=1 \\ i \neq m}}^{j} \frac{n}{p_i p_m}\,\phi(p_i p_m) \sum_{r \geq 1} \frac{1}{r}\left[\binom{n/p_i p_m - r - 1}{r - 1} + \binom{n/p_i^2 p_m - r - 1}{r - 1} + \cdots + \binom{n/p_i^{r_i} p_m^{r_m} - r - 1}{r - 1}\right] + \cdots$$

where the successive terms enumerate sequences having patterns of increasing multiplicity.

Let $y = (1 + \sqrt{5})/2$, $z = (1 - \sqrt{5})/2$. Then

$$nH(n) = (y^n + z^n + n - 1) + \sum_{i=1}^{j} \phi(p_i)\,\frac{n}{p_i}\left[\frac{p_i}{n}(y^{n/p_i} + z^{n/p_i} - 1) + \frac{p_i^2}{n}(y^{n/p_i^2} + z^{n/p_i^2} - 1) + \cdots + \frac{p_i^{r_i}}{n}(y^{n/p_i^{r_i}} + z^{n/p_i^{r_i}} - 1)\right]$$

$$+ \sum_{\substack{i,m=1 \\ i \neq m}}^{j} \phi(p_i p_m)\frac{n}{p_i p_m}\left[\frac{p_i p_m}{n}(y^{n/p_i p_m} + z^{n/p_i p_m} - 1) + \frac{p_i^2 p_m}{n}(y^{n/p_i^2 p_m} + z^{n/p_i^2 p_m} - 1) + \cdots + \frac{p_i^{r_i} p_m^{r_m}}{n}(y^{n/p_i^{r_i} p_m^{r_m}} + z^{n/p_i^{r_i} p_m^{r_m}} - 1)\right] + \cdots$$

$$= \text{I} + \text{II} + \text{III} + \cdots$$

Since $\phi(r)/r \le 1$ for any positive integer r, and $|z| < 1$, we have:

$$\text{II} \le \sum_{i=1}^{j} y^{n/p_i}(p_i + p_i^2 + \cdots + p_i^{r_i}) < \sum_{i=1}^{j} y^{n/2}2p_i^{r_i}$$

$$\le \sum_{i=1}^{j} y^{n/2}2n = \binom{j}{1}2ny^{n/2};$$

$$\text{III} \le \sum_{\substack{i,m=1\\ i\ne m}}^{j} y^{n/p_i p_m}(p_i + p_i^2 + \cdots + p_i^{r_i})(p_m + p_m^2 + \cdots + p_m^{r_m})$$

$$< \sum_{\substack{i,m=1\\ i\ne m}}^{j} y^{n/2}2p_i^{r_i}2p_m^{r_m} \le \sum_{\substack{i,m=1\\ i\ne m}}^{j} y^{n/2}4n = \binom{j}{2}4ny^{n/2}.$$

But for large n,

$$\sum_{i=0}^{j}\binom{j}{i}2^i ny^{n/2} \le \sum_{i=0}^{j}\binom{j}{i}n^2y^{n/2} = 2^j n^2 y^{n/2} \le n^3y^{n/2} = o(y^n).$$

So

$$nH(n) = y^n + o(y^n).$$

Similarly,

$$W(n) = \frac{1}{\sqrt{5}}(y^{n-1} - z^{n-1}) + \frac{1}{\sqrt{5}}\sum_{i=1}^{j}\phi(p_i)[(y^{n/p_i - 1} - z^{n/p_i - 1})$$

$$+ (y^{n/p_i^2 - 1} - z^{n/p_i^2 - 1}) + \cdots + (y^{n/p_i^{r_i} - 1} - z^{n/p_i^{r_i} - 1})]$$

$$+ \frac{1}{\sqrt{5}}\sum_{\substack{i,m=1\\ i\ne m}}^{j}\phi(p_i p_m)[(y^{n/p_i p_m - 1} - z^{n/p_i p_m - 1})$$

$$+ \cdots + (y^{n/p_i^{r_i}p_m^{r_m} - 1} - z^{n/p_i^{r_i}p_m^{r_m} - 1})] + \cdots = \frac{y^{n-1}}{\sqrt{5}} + o(y^n).$$

Thus, we have the following result on the asymptotic proportions of 1's in circular Fibonacci binary sequences.

$$\lim_{n\to\infty} Q(n) = (5 - \sqrt{5})/10.$$

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References

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3. J. Riordan. *Combinatorial Identities*. New York: Wiley, 1968.

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