ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

A. P. Hillman

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should include solutions.

Anyone desiring acknowledgment of contributions should enclose a stamped, self-addressed card (or envelope).

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-658 *Proposed by Joseph J. Kostal, U. of Illinois at Chicago*

Prove that $Q_1^2 + Q_2^2 + \cdots + Q_n^2 \equiv P_n^2 \pmod 2$, where the P_n and Q_n are the Pell numbers defined by

$$P_{n+2} = 2P_{n+1} + P_n,\ P_0 = 0,\ P_1 = 1;$$

$$Q_{n+2} = 2Q_{n+1} + Q_n,\ Q_0 = 1,\ Q_1 = 1.$$

B-659 *Proposed by Richard Andre-Jeannin, Sfax, Tunisia*

For $n \geq 3$, what is the nearest integer to $F_n\sqrt{5}$?

B-660 *Proposed by Herta T. Freitag, Roanoke, VA*

Find closed forms for:

$$\text{(i)}\quad 2^{1-n}\sum_{i=0}^{[n/2]}\binom{n}{2i}5^i,\quad \text{(ii)}\quad 2^{1-n}\sum_{i=1}^{[(n+1)/2]}\binom{n}{2i-1}5^{i-1},$$

where $[t]$ is the greatest integer in t.

B-661 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $T(n) = n(n + 1)/2$. In B-646, it was seen that $T(n)$ is an integral divisor of $T(2T(n))$ for all n in $Z^+ = \{1, 2, \ldots\}$. Find the n in Z^+ such that $T(n)$ is an integral divisor of

$$\sum_{i=1}^{n} T(2T(i)).$$

B-662 *Proposed by H.-J. Seiffert, Berlin, Germany*

Let $H_n = L_n P_n$, where the L_n and P_n are the Lucas and Pell numbers, respectively. Prove the following congruences modulo 9:

(1) $H_{4n} \equiv 3n$, (2) $H_{4n+1} \equiv 3n + 1$,

(3) $H_{4n+2} \equiv 3n + 6$, (4) $H_{4n+3} \equiv 3n + 2$.

B-663 *Proposed by Clark Kimberling, U. of Evansville, Evansville, IN*

Let $t_1 = 1$, $t_2 = 2$, and $t_n = (3/2)t_{n-1} - t_{n-2}$ for $n = 3, 4, \ldots$. Determine lim sup t_n.

SOLUTIONS

When Is $2^n \equiv n \pmod 5$?

B-634 *Proposed by P. L. Mana, Albuquerque, NM*

For how many integers n with $1 \le n \le 10^6$ is $2^n \equiv n \pmod 5$?

Solution by Hans Kappus, Rodersdorf, Switzerland

More generally, we show that the number of solutions of

$$2^n \equiv n \pmod 5 \tag{*}$$

with $1 \le n \le 10^r$ is $2 \cdot 10^{r-1}$. In fact, it is easily checked that $(2^n - n)$ mod 5 is periodic with period $p = 20$ since $p = 20$ is the smallest number such that $2^n(2^p - 1) \equiv p \pmod 5$ for all $n \in N$. Now the only solutions of (*) with $1 \le n \le 20$ are n = 3, 14, 16, 17. Hence, the number of solutions of (*) in the interval $[1, 10^r]$ is $4 \cdot 10^r/20 = 2 \cdot 10^{r-1}$.

Also solved by R. Andre-Jeannin, Charles Ashbacher, Paul S. Bruckman, John Cannell, Nickolas D. Diamantis, Alberto Facchini, Piero Filipponi, Russell Jay Hendel, H. Klauser & M. Wachtel, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Sahib Singh, Lawrence Somer, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Application of the Inequality on the Means

B-635 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

For all positive integers n, prove that

$$2^{n+1}\left[1 + \sum_{k=1}^{n} (k!k)\right] < (n + 2)^{n+1}.$$

Solution by Bob Prielipp, U. of Wisconsin-Oshkosh

$$\sum_{k=1}^{n}(k!k) = \sum_{k=1}^{n}((k+1)! - k!) = (n+1)! - 1$$

Thus, the required inequality is equivalent to

$$(n+1)! < \left(\frac{n+2}{2}\right)^{n+1}.$$

This inequality follows immediately from the Arithmetic Mean-Geometric Mean Inequality, since

$$\frac{1 + 2 + \cdots + n + (n+1)}{n+1} = \frac{(n+1)(n+2)}{2(n+1)} = \frac{n+2}{2}.$$

Also solved by R. Andre-Jeannin, Charles Ashbacher, Paul S. Bruckman, J. E. Chance, Nicholas D. Diamantis, Russell Euler, Piero Filipponi, Hans Kappus, Y. H. Harris Kwong, Carl Libis, Alejandro Necochea, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Difference Equation

B-636 *Proposed by Mohammad K. Azarian, U. of Evansville, Evansville, IN*

Solve the difference equation

$$x_{n+1} = (n+1)x_n + \lambda(n+1)^3[n!(n! - 1)]$$

for x_n in terms of λ, x_0, and n.

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

Divide the recurrence relation by $(n+1)!$:

$$\frac{x_{n+1}}{(n+1)!} = \frac{x_n}{n!} + \lambda[(n+1)(n+1)! - (n+1)^2].$$

Let $a_n = x_n/n!$. We then have

$$a_{n+1} = a_n + \lambda[(n+1)(n+1)! - (n+1)^2] \qquad \text{for } n \geq 0,$$

from which it follows immediately that

$$a_n = a_0 + \lambda\sum_{k=1}^{n}(k!k - k^2).$$

Since $\sum_{k=0}^{n} k!k = (n+1)! - 1$, $\sum_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$, and $a_0 = x_0$, we obtain

$$x_n = n!\{x_0 + \lambda[(n+1)! - 1 - n(n+1)(2n+1)/6]\}.$$

Also solved by R. Andre-Jeannin, Paul S. Bruckman, Nicholas D. Diamantis, Guo-Gang Gao, Hans Kappus, L. Kuipers, H.-J. Seiffert, Amitabha Tripathi, and the proposer.

Golden Geometric Series

B-637 *Proposed by John Turner, U. of Waikato, Hamilton, New Zealand*

Show that

$$\sum_{n=1}^{\infty} \frac{1}{F_n + aF_{n+1}} = 1,$$

where a is the golden mean $(1 + \sqrt{5})/2$.

Solution by Sahib Singh, Clarion U. of Pennsylvania, Clarion, PA

By induction, $F_n + aF_{n+1} = a^{n+1}$. Thus, the given sum equals

$$\sum_{n=1}^{\infty} \frac{1}{a^{n+1}}.$$

Since $1/a < 1$, the sum of this geometric series is

$$\frac{1/a^2}{1 - (1/a)} = \frac{1}{a(a - 1)} = \frac{1}{1} = 1$$

Also solved by R. Andre-Jeannin, Paul S. Bruckman, John Cannell, J. E. Chance, Nickolas D. Diamantis, Russell Euler, Piero Filipponi, Herta T. Freitag, Guo-Gang Gao, Russell Jay Hendel, Hans Kappus, Joseph J. Kostal, L. Kuipers, Y. H. Harris Kwong, Carl Libis, Alejandro Necochea, Oxford Running Club (U. of Mississippi), Bob Prielipp, Elmer D. Robinson, H.-J. Seiffert, A. G. Shannon, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Summing Every Fourth Fibonacci Number

B-638 *Proposed by Herta T. Freitag, Roanoke, VA*

Find s and t as function of k and n such that

$$\sum_{i=1}^{k} F_{n-4k+4i-2} = F_s F_t.$$

Solution by Paul S. Bruckman, Edmonds, WA

$$\sum_{i=1}^{k} F_{n-4k+4i-2} = \sum_{i=0}^{k-1} F_{n-4i-2} = \frac{1}{5}\sum_{i=0}^{k-1}(L_{n-4i} - L_{n-4i-4})$$
$$= \frac{1}{5}(L_n - L_{n-4k}) = F_{2k}F_{n-2k}.$$

Hence, we may take $s = 2k$, $t = n - 2k$ (or $s = n - 2k$, $t = 2k$).

Also solved by R. Andre-Jeannin, Piero Filipponi, Russell Jay Hendel, L. Kuipers, Y. H. Harris Kwong, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

Lucas Analogue

B-639 *Proposed by Herta T. Freitag, Roanoke, VA*

Find s and t as functions of k and n such that

$$\sum_{i=1}^{k} L_{n-4k+4i-2} = F_s L_t.$$

Solution by Y. H. Harris Kwong, SUNY College at Fredonia, Fredonia, NY

It is well known that $L_n = a^n + b^n$, where a and b are the zeros of $x^2 - x - 1$; so we can employ the same technique used in solving B-638. Alternately, using the result (from B-638)

$$\sum_{i=1}^{k} F_{n-4k+4i-2} = F_{2k}F_{n-2k},$$

and the fact that $L_n = F_{n+1} - F_{n-1}$, a solution follows immediately:

$$\sum_{i=1}^{k} L_{n-4k+4i-2} = \sum_{i=1}^{k} F_{n+1-4k+4i-2} + \sum_{i=1}^{k} F_{n-1-4k+4i-2}$$

$$= F_{2k}F_{n+1-2k} + F_{2k}F_{n-1-2k}$$

$$= F_{2k}L_{n-2k}.$$

Also solved by Paul S. Bruckman, R. Andre-Jeannin, Piero Filipponi, L. Kuipers, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Amitabha Tripathi, Gregory Wulczyn, and the proposer.

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