ON THE FIBONACCI NUMBER OF AN $M \times N$ LATTICE

Konrad Engel
Wilhelm-Pieck-Universität, Sektion Mathematik, 2500 Rostock, German Democratic Republic
(Submitted March 1988)

1. Introduction

Let

$$Z_{m,n} := \{(i, j): 1 \le i \le m,\ 1 \le j \le n\},$$

$$\mathscr{A}_{m,n} := \{A \subseteq Z_{m,n}: \text{ there are no } (i_1, j_1), (i_2, j_2) \in A \text{ with } |i_1 - i_2| + |j_1 - j_2| = 1\}$$

and

$$\kappa_{m,n} := |\mathscr{A}_{m,n}|.$$

So $\kappa_{m,n}$ equals the number of independent (vertex) sets in the Hasse graph of a product of two chains with m resp. n elements, i.e., in the $m \times n$ lattice. Following Prodinger & Tichy [11], we call $\kappa_{m,n}$ the Fibonacci number of the $m \times n$ lattice.

In this paper we study the numbers $\kappa_{m,n}$ using linear algebraic techniques. We prove several inequalities for these numbers and show that

$$1.503 \le \lim_{n \to \infty} \kappa_{n,n}^{1/n^2} \le 1.514.$$

We conjecture that this limit equals 1.50304808... .

The problem of the determination of the number of independent sets in graphs goes back to Kaplansky [6] who determined in his well-known lemma the number of k-element independent sets in the $1 \times n$ lattice, i.e., in a path on n vertices. Burosch suggested to consider other graphs, and some results were obtained in [3].

Answering a question of Weber, the number of independent sets in the Hasse graph of the Boolean lattice was determined asymptotically by Korshunov & Saposhenko [9]. Prodinger & Tichy [11] and later together with Kirschenhofer [7], [8] considered that problem in particular for trees. They introduced the notion of the Fibonacci number of a graph for the number of independent sets in it because the case of paths yields the Fibonacci numbers. We will see that the numbers $\kappa_{m,n}$ preserve many properties of the classical Fibonacci numbers, i.e., the results do not hold only for $m = 1$ but for all positive integers m. The first results on the numbers $\kappa_{m,n}$ have been obtained by Weber [12]. Among other things he proved the inequality

$$1.45^{mn} < \kappa_{m,n} < 1.74^{mn} \text{ if } mn > 1,$$

the existence of

$$\lim_{n \to \infty} \kappa_{m,n}^{1/n} \quad \text{and} \quad \lim_{n \to \infty} \kappa_{n,n}^{1/n^2}$$

as well as the inequality

$$1.45 \le \lim_{n \to \infty} \kappa_{n,n}^{1/n^2} \le 1.554.$$

2. Inequalities and Eigenvalues

Let

$$\varphi_m := \{S \subseteq \{1, \ldots, m\}: \text{there are no } i, j \in S \text{ with } |i - j| = 1\},$$
$$\mathscr{A}_{m,n,S} := \{A \in \mathscr{A}_{m,n}: (i, n) \in A \text{ iff } i \in S\},$$
$$x_{m,n,S} := |\mathscr{A}_{m,n,S}|.$$

So $x_{m,n,S}$ counts those sets of $\mathscr{A}_{m,n}$ for which the elements in the top line (with second coordinate n) are fixed by S. Obviously, $|\varphi_m| = \kappa_{m,1}$. Briefly, we set $z_m := \kappa_{m,1}$.

Throughout this section we consider m to be fixed. To avoid too many indices, we omit the index m everywhere. Obviously,

(1) $\kappa_n = x_{n+1,\phi}$.

Moreover,

(2) $x_{n+1,S} = \sum_{\substack{T \in \varphi \\ T \cap S = \phi}} x_{n,T}$ for all $S \in \varphi$, $n = 1, 2, \ldots$.

Let $\mathbf{x}_n$ be the vector whose coordinates are the numbers $x_{n,S}$ ($S \in \varphi$) and $A = (a_{S,T})_{S,T \in \varphi}$ that $z \times z$-matrix for which

$$a_{S,T} := \begin{cases} 1 \text{ if } S \cap T = \phi, \\ 0 \text{ otherwise.} \end{cases}$$

Because of (2), we have

(3) $\mathbf{x}_{n+1} = A\mathbf{x}_n$, $n = 1, 2, \ldots$.

Let the vector $\mathbf{e}$ with coordinates e_S, $S \in \varphi$, be defined by

$$e_S := \begin{cases} 1 \text{ if } S = \phi, \\ 0 \text{ otherwise,} \end{cases}$$

and let, for an integer k, the vector $\mathbf{k}$ be composed only of k's. Then we have

(4) $\mathbf{x}_1 = A\mathbf{e} = \mathbf{1}$,

and because of (3),

(5) $\mathbf{x}_n = A^n\mathbf{e}$.

Finally, if (,) denotes the inner product, then

(6) $\kappa_n = x_{n+1,\phi} = (A^{n+1}\mathbf{e}, \mathbf{e})$.

In our proofs, we often use the fact that A is symmetric. In particular, we have, for all vectors $\mathbf{x}$, $\mathbf{y}$,

(7) $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A\mathbf{y})$.

Theorem 1: For all positive integers k and ℓ,

(8) $\kappa_{k+1}^2 \le \kappa_{2k-1}\kappa_{2\ell+1}$.

Proof: By (6), (7), and the Cauchy-Schwarz inequality, we have

$$\kappa_{k+1}^2 = (A^{k+\ell+1}\mathbf{e}, \mathbf{e})^2 = (A^k\mathbf{e}, A^{\ell+1}\mathbf{e})^2 \le (A^k\mathbf{e}, A^k\mathbf{e})(A^{\ell+1}\mathbf{e}, A^{\ell+1}\mathbf{e})$$
$$= (A^{2k}\mathbf{e}, \mathbf{e})(A^{2\ell+2}\mathbf{e}, \mathbf{e}) = \kappa_{2k-1}\kappa_{2\ell+1}. \quad \square$$

Corollary 2: $\frac{\kappa_3}{\kappa_1} \le \frac{\kappa_5}{\kappa_3} \le \frac{\kappa_7}{\kappa_5} \le \cdots$. □

Since A is symmetric, all eigenvalues of A are real numbers. Let λ be the largest eigenvalue of A.

Proposition 3: λ has multiplicity 1, to λ belongs an eigenvector $\mathbf{u}$ with coordinates $u_S > 0$ for all $S \in \varphi$, and $|\lambda| > |\mu|$ for all eigenvalues μ of A.

Proof: The column and row of A which correspond to the empty set ϕ contain only ones; hence, the matrix A is irreducible and A^2 is positive (see [4], p. 395). Now the statements in the proposition are direct consequences of two theorems of Frobenius (see [4], pp. 398, 422). □

Theorem 4: Let $\mathbf{u}$ be that eigenvector of A to the largest eigenvalue λ for which $u_S > 0$ for all $S \in \varphi$ and $\sum_{S \in \varphi} u_S^2 = 1$ holds. Then

$$x_{n,S} \sim u_\phi u_S \lambda^n \text{ as } n \to \infty.$$

Proof: We use standard techniques. Let U be the orthogonal matrix whose columns are normed, pairwise orthogonal eigenvectors of A and let D be the diagonal matrix of the corresponding eigenvalues. Then

$$U^T A U = D \quad \text{and} \quad A^n = U D^n U^T.$$

Consequently,

$$\mathbf{x}_n = A^n \mathbf{e} = U D^n U^T \mathbf{e} \text{ [note (5)].}$$

Because of Proposition 3, the asymptotic behavior of the components of $\mathbf{x}_n$ is determined by the terms containing λ^n which yields the formula in the theorem. □

Noting (1) and Corollary 2, we derive immediately

Corollary 5:

(a) $\lim_{n \to \infty} \kappa^{1/n} = \lambda$, (b) $\lim_{n \to \infty} \frac{\kappa_{n+1}}{\kappa_n} = \lambda$,

(c) $\lim_{n \to \infty} \frac{\kappa_{n+2}}{\kappa_n} = \lambda^2$, (d) $\frac{\kappa_{2k+1}}{\kappa_{2k-1}} \le \lambda^2$ for all $k = 1, 2, \ldots$. □

Remarks:

(a) If $p(\mu) = \det(\mu E - A) = \mu^z + a_{z-1}\mu^{z-1} + \cdots + a_0$ is the characteristic polynomial of A, we have $a_{z-1} = -\text{trace } A = -1$ and (by induction)

$$|a_0| = |\det A| = 1.$$

From the Cayley-Hamilton relation, it follows that

$$\mathbf{x}_{n+z} = -a_{z-1}\mathbf{x}_{n+z-1} - \cdots - a_0 \mathbf{x}_n$$

and, in particular, the recursion

$$\kappa_{n+z} = -a_{z-1}\kappa_{n+z-1} - \cdots - a_0 \kappa_n.$$

(b) Corollary 5(b) contains in effect the crucial point of the well-known power method of *v.* Mises for the determination of the absolute maximal eigenvalue of a matrix.

Theorem 6: For all positive integers h, k, ℓ,

$$(\kappa_{h+2\ell-1}/\kappa_{2\ell-1})^{1/h} \le \lambda \le \kappa_k^{1/k}.$$

Proof: It is well known that the Rayleigh-Quotient does not exceed the largest eigenvalue. Hence, by (6) and (7),

$$\begin{aligned}\kappa_{h+2\ell-1}/\kappa_{2\ell-1} &= (A^{h+2\ell}\mathrm{e}, \mathrm{e})/(A^{2\ell}\mathrm{e}, \mathrm{e}) \\ &= (A^h(A^\ell \mathrm{e}), A^\ell \mathrm{e})/(A^\ell \mathrm{e}, A^\ell \mathrm{e}) \\ &\le \text{largest eigenvalue of } A^h = \lambda^h.\end{aligned}$$

This proves the left inequality.

To show the right inequality, we use a standard technique for the estimation of the largest eigenvalue of nonnegative matrices (see, e.g., [10], 11.14). Let $\mathbf{u}$ be the eigenvector of A to λ with $u_S > 0$ for all $S \in \varphi$ and with $u_\phi = 1$. Then $A\mathbf{u} = \lambda\mathbf{u}$ implies

$$\sum_{T\in\varphi} u_T = \lambda u_\phi = \lambda, \qquad \sum_{\substack{T\in\varphi \\ T\cap S=\phi}} u_T = \lambda u_S,\ S \in \varphi.$$

Hence, $1 \ge u_S$ for all $S \in \varphi$ and, consequently,

$$\mathbf{u} \le \mathbf{1}.$$

It follows [note (4)] that

$$\lambda^k \mathbf{u} = A^k \mathbf{u} \le A^k \mathbf{1} = A^{k+1}\mathrm{e},$$

which gives [note (6)]

$$\lambda^k = (\lambda^k \mathbf{u}, \mathrm{e}) \le (A^{k+1}\mathrm{e}, \mathrm{e}) = \kappa_k,$$

i.e., the right inequality. □

Corollary 7: For all positive integers ℓ, k with $k > 2\ell - 1$,

$$\kappa_k^{1/k} \le \kappa_{2\ell-1}^{1/(2\ell-1)}.$$

Proof: We choose $h := k - (2\ell - 1)$ in Theorem 6. Then

$$(\kappa_k/\kappa_{2\ell-1})^{1/h} \le \kappa_k^{1/k}$$

and, equivalently,

$$\kappa_k^k/\kappa_{2\ell-1}^k \le \kappa_k^h, \qquad \kappa_k^{2\ell-1} \le \kappa_{2\ell-1}^k. \ \square$$

3. Limits

Now we consider the dependence of m and introduce again everywhere the index m. We will study the sequence $\{\lambda_m^{1/m}\}$, where

$$\lambda_m = \text{largest eigenvalue of } A_m = \lim_{n\to\infty} \kappa_{m,n}^{1/n} \text{ [see Corollary 5(a)].}$$

In the following, we often use the obvious fact that

(9) $\quad \kappa_{m,n} = \kappa_{n,m}$ for all n, m.

Proposition 8: For all integers ℓ, k with $k > 2\ell - 1$,

$$\lambda_k^{1/k} \le \lambda_{2\ell-1}^{1/(2\ell-1)}.$$

Proof: By Corollary 7 and (9), we have

$$\kappa_{k,m}^{1/k} \le \kappa_{2\ell-1,m}^{1/(2\ell-1)}$$

and further

$$(\kappa_{k,m}^{1/m})^{1/k} \le (\kappa_{2\ell-1,m}^{1/m})^{1/(2\ell-1)}.$$

Now, if m tends to infinity, we obtain

$$\lambda_k^{1/k} \le \lambda_{2\ell-1}^{1/(2\ell-1)}. \quad \square$$

Proposition 9: The limit $g := \lim_{m\to\infty} \lambda_m^{1/m}$ exists, and

$$\lambda_{2\ell}/\lambda_{2\ell-1} \le g \le \lambda_k^{1/k}$$

holds for all positive integers ℓ and k.

Proof: First we note that the existence of the limit is trivial if Conjectures 2 and 3 are true, because then the sequence $\{\lambda_m^{1/m}\}$ is monotoniously decreasing. Let

$$\gamma := \liminf_{n\to\infty} \lambda_m^{1/m} \quad \text{(note Proposition 8)}.$$

Now choose $\varepsilon > 0$ and let M be a number that satisfies

$$(10) \qquad \lambda_M^{1/M} < \gamma + \varepsilon/2.$$

Because M is fixed, by Corollary 5(a) there is a number m_0 such that, for all $m > m_0$,

$$(11) \qquad (\kappa_{M,m}^{1/m})^{1/M} < \lambda_M^{1/M} + \varepsilon/2.$$

Finally, by Theorem 6,

$$(12) \qquad \lambda_m \le \kappa_{m,M}^{1/M} \text{ for all } m = 1, 2, \ldots .$$

From (10), (11), and (12), we derive, for all $m > m_0$,

$$\lambda_m^{1/m} \le (\kappa_{m,M}^{1/m})^{1/M} < \lambda_M^{1/M} + \varepsilon/2 < \gamma + \varepsilon.$$

Consequently,

$$g = \lim_{m\to\infty} \lambda_m^{1/m} = \gamma.$$

Last, but not least, again by Theorem 6 (with $h = 1$),

$$(\kappa_{2\ell,m}/\kappa_{2\ell-1,m})^{1/m} = (\kappa_{m,2\ell}/\kappa_{m,2\ell-1})^{1/m} \le \lambda_m^{1/m} \le (\kappa_{m,k}^{1/k})^{1/m} = (\kappa_{k,m}^{1/m})^{1/k},$$

and with $m \to \infty$, we obtain

$$\lambda_{2\ell}/\lambda_{2\ell-1} \le g \le \lambda_k^{1/k}. \quad \square$$

Theorem 10: The limit $\lim_{n\to\infty} \kappa_{n,n}^{1/n^2}$ exists, and it is equal to g. In particular,

$$\lambda_{2\ell}/\lambda_{2\ell-1} \le \lim_{n\to\infty} \kappa_{n,n}^{1/n^2} \le \lambda_k^{1/k}$$

for all positive integers k and ℓ.

Proof: By Theorem 6 (with $h = k = m = n$ and $\ell = 1$) and using the obvious fact that $\kappa_{n,n} \le \kappa_{n,n+1}$,

$$(\kappa_{n,n}/\kappa_{n,1})^{1/n} \le (\kappa_{n,n+1}/\kappa_{n,1})^{1/n} \le \lambda_n \le \kappa_{n,n}^{1/n}.$$

Hence,

$$\lambda_n^{1/n} \le \kappa_{n,n}^{1/n^2} \le (\kappa_{n,1}^{1/n})^{1/n} \lambda_n^{1/n}.$$

If $n \to \infty$, then the lower and upper bounds tend to g, by Corollary 5(a) and Proposition 9; hence, $\kappa_{n,n}^{1/n^2}$ also tends to g. The inequality in this theorem is a reformulation of the inequality in Proposition 9. $\square$

We note here that the existence of the limit in Theorem 10 was previously proved by Weber (see [12]).

To find bounds for $\lim_{n\to\infty} \kappa_{n,n}^{1/n^2}$, we used a computer (see Table 1).

TABLE 1

m	λ_m/λ_{m-1}	$\lambda_m^{1/m}$
2	1.49206604	1.55377397
3	1.50416737	1.53705928
4	1.50292823	1.52845453
5	1.50306010	1.52334155
6	1.50304676	1.51994015
7	1.50304821	1.51751544
8	1.50304807	1.51569943
9	1.50304808	1.51428849
10	1.50304808	1.51316067

Because of Theorem 10 and the numerical results, we have the following estimation.

Corollary 11: $1.50304808 \le \lim_{n\to\infty} \kappa_{n,n}^{1/n^2} \le 1.51316067.$ □

Conjecture 1: For all positive integers m and ℓ,

$$\kappa_{m,\,2\ell+1}/\kappa_{m,\,2\ell} \ge \lambda_m.$$

If this conjecture is true, then it would follow, as above, that

$$\lim_{n\to\infty} \kappa_{n,n}^{1/n^2} \le \lambda_{2\ell+1}/\lambda_{2\ell};$$

hence (with $\ell = 4$),

$$\lim_{n\to\infty} \kappa_{n,n}^{1/n^2} = 1.50304808\ldots .$$

Let us note that, for numerical purposes, the bound $\lambda_m^{1/m}$ is weak, because $\lambda_m^{1/m}$ decreases slowly whereas the size of the matrix A_m increases exponentially with m (like the Fibonacci numbers). The following conjecture is stronger.

Conjecture 2: For all positive integers m and k,

$$\kappa_{m,\,2k}^2 \ge \kappa_{m,\,2k-2}\kappa_{m,\,2k+2}.$$

If this Conjecture is true, then, together with Theorem 1 and Corollary 5, it would follow (we omit again the index m)

$$\frac{\kappa_3}{\kappa_1} \le \frac{\kappa_5}{\kappa_3} \le \frac{\kappa_7}{\kappa_5} \le \cdots \le \lambda^2 \le \cdots \le \frac{\kappa_6}{\kappa_4} \le \frac{\kappa_4}{\kappa_2} \le \frac{\kappa_2}{\kappa_0}$$

and, further,

$$\frac{\kappa_2}{\kappa_1} \le \frac{\kappa_4}{\kappa_3} \le \frac{\kappa_6}{\kappa_5} \le \cdots \le \lambda \le \cdots \le \frac{\kappa_5}{\kappa_4} \le \frac{\kappa_3}{\kappa_2} \le \frac{\kappa_1}{\kappa_0}.$$

Conjecture 3: For all positive integers m and k,

$$(\kappa_{m,2k+1}/\kappa_{m,2k})^2 \leq \kappa_{m,2k}/\kappa_{m,2k-2}.$$

If Conjectures 2 and 3 are true, then one can derive (again without index m)

$$(\kappa_{2k+1}/\kappa_{2k})^{2k} \leq (\kappa_{2k}/\kappa_{2k-2})^k \leq \frac{\kappa_{2k}}{\kappa_{2k-2}} \frac{\kappa_{2k-2}}{\kappa_{2k-4}} \cdots \frac{\kappa_2}{\kappa_0} = x_{2k},$$

i.e.,

$$\kappa_{2k+1}^{2k} \leq \kappa_{2k}^{2k+1}, \quad \kappa_{2k+1}^{1/(2k+1)} \leq \kappa_{2k}^{1/2k},$$

and, together with Corollary 7, this means that the sequence $\{\kappa_{m,n}^{1/n}\}$ decreases monotoniously in n. Finally, as in the proof of Proposition 8, one can conclude that $\{\lambda_m^{1/m}\}$ decreases monotoniously in m.

Because of the recursions

$$\kappa_{1,n+2} = \kappa_{1,n+1} + \kappa_{1,n} \quad \text{and} \quad \kappa_{2,n+2} = 2\kappa_{2,n+1} + \kappa_{2,n},$$

one can easily verify these conjectures for $m = 1, 2$ (see also [2]). Using a computer, we verified them also for the numbers $\kappa_{m,n}$ for which $3 \leq m \leq 10$ and $1 \leq n \leq 20$.

References

1. D. M. Cvetković, M. Doob, & H. Sachs. *Spectra of Graphs.* Berlin: VEB Deutscher Verlag der Wissenschaften, 1980.
2. T. P. Dence. "Ratios of Generalized Fibonacci Sequences." *Fibonacci Quarterly 25.2* (1987):137-143.
3. K. Engel. "Über zwei Lemmata von Kaplansky." *Rostock. Math. Kolloq. 9* (1978):5-26.
4. F. R. Gantmacher. *Matrizentheorie.* Berlin: VEB Deutscher Verlag der Wissenschaften, 1980.
5. G. Hopkins & W. Staton. "Some Identities Arising from the Fibonacci Numbers of Certain Graphs." *Fibonacci Quarterly 22.3* (1984):255-258.
6. I. Kaplansky. "Solution of the 'Problème des ménages.'" *Bull. Amer. Math. Soc. 49* (1943):784-785.
7. P. Kirschenhofer, H. Prodinger, & R. F. Tichy. "Fibonacci Numbers of Graphs II." *Fibonacci Quarterly 21.3* (1983):219-229.
8. P. Kirschenhofer, H. Prodinger, & R. F. Tichy. "Fibonacci Numbers and Their Applications." In *Fibonacci Numbers and Their Applications* (Proc. 1st Int. Conf., Patras, Greece, 1984). Dordrecht, Holland: D. Reidel, 1986, pp. 105-120.
9. A. D. Korshunov & A. A. Saposhenko. "On the Number of Codes with Distance 2*T*." *Problemy Kibernet. 40* (1983):111-130.
10. L. Lovász. *Combinatorial Problems and Exercises.* Budapest: Akadémiai Kiadó, 1979.
11. H. Prodinger & R. F. Tichy. "Fibonacci Numbers of Graphs." *Fibonacci Quarterly 20.1* (1982):16-21.
12. K. Weber. "On the Number of Stable Sets in an $m \times n$ Lattice." *Rostock. Math. Kolloq. 34* (1988):28-36.

*****