

LAMBERT SERIES AND THE SUMMATION OF RECIPROALS  
IN CERTAIN FIBONACCI-LUCAS-TYPE SEQUENCES

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1. Introduction

Consider the sequence of real numbers defined by the recurrence relation

$$(1.1) \quad W_n = pW_{n-1} + W_{n-2},$$

where  $p$  is a strictly positive real number. Special cases of  $(W_n)$  which interest us here are:

$$(1.2) \quad U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (\text{Fibonacci-type sequence}),$$

and

$$(1.3) \quad V_n = \alpha^n + \beta^n \quad (\text{Lucas-type sequence}),$$

where

$$(1.4) \quad \alpha = \frac{p + \sqrt{p^2 + 4}}{2},$$

$$\beta = \frac{p - \sqrt{p^2 + 4}}{2}.$$

It is clear that

$$(1.5) \quad \alpha\beta = -1, \quad \alpha > 1, \quad -1 < \beta < 0.$$

On the other hand, the Lambert series is defined by

$$(1.6) \quad L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}, \quad |x| < 1.$$

It has been known for a long time (see Horadam [1] for complete references) that

$$\sum_{n=1}^{\infty} \frac{1}{U_{2n}} = (\alpha - \beta)[L(\beta^2) - L(\beta^4)],$$

$$\sum_{n=1}^{\infty} \frac{1}{V_{2n-1}} = -L(\beta) + 2L(\beta^2) - L(\beta^4).$$

The purpose of this paper is to establish the following result.

*Theorem 1:*

$$(1.7) \quad \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+1}} = 2(\alpha - \beta)[L(\beta^2) - 2L(\beta^4) + 2L(\beta^8)] + \beta;$$

$$(1.8) \quad \sum_{n=1}^{\infty} \frac{1}{V_n V_{n+1}} = \frac{2}{\alpha - \beta}[L(\beta^2) - 2L(\beta^8)] + \frac{\beta}{(\alpha - \beta)p}.$$

2. Preliminary Lemma

*Lemma 1:*

$$(2.1) \quad \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 - x^{2n+1}} = L(x) - L(x^2);$$

$$(2.2) \quad \sum_{n=1}^{\infty} \frac{x^n}{1+x^n} = L(x) - 2L(x^2);$$

$$(2.3) \quad \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1+x^{2n+1}} = L(x) - 3L(x^2) + 2L(x^4).$$

(2.1) is obviously true, whereas (2.2) follows from the identity

$$\frac{x^n}{1+x^n} = \frac{x^n}{1-x^n} - \frac{2x^{2n}}{1-x^{2n}},$$

and (2.3) follows from

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{1+x^{2n+1}} = \sum_{n=1}^{\infty} \frac{x^n}{1+x^n} - \sum_{n=1}^{\infty} \frac{x^{2n}}{1+x^{2n}}.$$

### 3. Proof of Theorem 1

Lemma 2:

$$(3.1) \quad 2 \sum_{n=1}^{\infty} \frac{1}{\alpha^n U_n} = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+1}};$$

$$(3.2) \quad 2 \sum_{n=1}^{\infty} \frac{1}{\alpha^n V_n} = \frac{1}{\alpha p} + \sum_{n=1}^{\infty} \frac{\alpha - \beta}{V_n V_{n+1}}$$

Proof: First, we have

$$\begin{aligned} \alpha U_{n+1} + U_n &= \frac{1}{\alpha - \beta} \left[ \alpha \left( \alpha^{n+1} - (-1)^{n+1} \frac{1}{\alpha^{n+1}} \right) + \alpha^n - (-1)^n \frac{1}{\alpha^n} \right] \\ &= \frac{1}{\alpha - \beta} (\alpha^{n+2} + \alpha^n) = \frac{\alpha^{n+1}}{\alpha - \beta} \left( \alpha + \frac{1}{\alpha} \right) = \alpha^{n+1}. \end{aligned}$$

Thus,

$$\frac{1}{\alpha^n U_n} + \frac{1}{\alpha^{n+1} U_{n+1}} = \frac{1}{U_n U_{n+1}}.$$

By adding this term by term, we find (3.1) since  $U_1 = 1$ . The proof of (3.2) follows the same pattern if we observe that

$$\alpha V_{n+1} + V_n = (\alpha - \beta) \alpha^{n+1}.$$

Thus,

$$\frac{1}{\alpha^n V_n} + \frac{1}{\alpha^{n+1} V_{n+1}} = \frac{\alpha - \beta}{V_n V_{n+1}}.$$

Now, adding this term by term, we find (3.2) since  $V_1 = p$ .

Lemma 3:

$$(3.3) \quad \sum_{n=1}^{\infty} \frac{1}{\alpha^n U_n} = (\alpha - \beta) [L(\beta^2) - 2L(\beta^4) + 2L(\beta^8)];$$

$$(3.4) \quad \sum_{n=1}^{\infty} \frac{1}{\alpha^n V_n} = L(\beta^2) - 2L(\beta^8).$$

Proof:

$$\begin{aligned} \frac{1}{\alpha - \beta} \sum_{n=1}^{\infty} \frac{1}{\alpha^n U_n} &= \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n} - (-1)^n} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1 - (-1)^n \beta^{2n}} \\ &= \sum_{n=1}^{\infty} \frac{\beta^{4n}}{1 - \beta^{4n}} + \sum_{n=0}^{\infty} \frac{\beta^{4n+2}}{1 + \beta^{4n+2}} \end{aligned}$$

Using (1.6) with  $x = \beta^4$  and (2.3) with  $x = \beta^2$ , we find (3.3). On the other hand, we have:

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^n V_n} = \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n} + (-1)^n} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1 + (-1)^n \beta^{2n}} = \sum_{n=1}^{\infty} \frac{\beta^{4n}}{1 + \beta^{4n}} + \sum_{n=0}^{\infty} \frac{\beta^{4n+2}}{1 - \beta^{4n+2}}.$$

Using (2.2) with  $x = \beta^4$  and (2.1) with  $x = \beta^2$ , we find (3.4). This concludes the proof of Lemma 3. Now the proof of the theorem follows immediately from Lemmas 2 and 3.

#### 4. Special Cases

##### 4.1 Fibonacci-Lucas Sequences

Let  $p = 1$  in (1.1) to obtain

$$W_n = W_{n-1} + W_{n-2}, \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

$U_n = F_n$  is the Fibonacci sequence and  $V_n = L_n$  is the Lucas sequence. Equations (1.7) and (1.8) take the following form:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} &= 2\sqrt{5} \left[ L\left(\frac{3 - \sqrt{5}}{2}\right) - 2L\left(\frac{7 - 3\sqrt{5}}{2}\right) + 2L\left(\frac{47 - 21\sqrt{5}}{2}\right) \right] + \frac{1 - \sqrt{5}}{2}; \\ \sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1}} &= \frac{2}{\sqrt{5}} \left[ L\left(\frac{3 - \sqrt{5}}{2}\right) - 2L\left(\frac{47 - 21\sqrt{5}}{2}\right) \right] + \frac{1 - \sqrt{5}}{2\sqrt{5}}. \end{aligned}$$

##### 4.2 Pell and Pell-Lucas Sequences

Let  $p = 2$  in (1.1) to obtain

$$W_n = 2W_{n-1} + W_{n-2}, \quad \alpha = 1 + \sqrt{2}, \quad \beta = 1 - \sqrt{2}.$$

$U_n = P_n$  is the Pell sequence,  $V_n = Q_n$  is the Pell-Lucas sequence. Equations (1.7) and (1.8) take the form:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{P_n P_{n+1}} &= 4\sqrt{2} [L(3 - 2\sqrt{2}) - 2L(17 - 12\sqrt{2}) + 2L(577 - 408\sqrt{2})] + 1 - \sqrt{2}; \\ \sum_{n=1}^{\infty} \frac{1}{Q_n Q_{n+1}} &= \frac{1}{\sqrt{2}} [L(3 - 2\sqrt{2}) - 2L(577 - 408\sqrt{2})] + \frac{1 - \sqrt{2}}{4\sqrt{2}}. \end{aligned}$$

#### 5. Generalization

The following theorem generalizes the above result. It is given without proof, since the methods required exactly parallel those of Section 3. We assume that  $K$  is an odd integer.

*Theorem 2:*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{U_{kn} U_{k(n+1)}} &= \frac{2(\alpha - \beta)}{U_k} [L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})] + \frac{\beta^k}{U_k^2}; \\ \sum_{n=1}^{\infty} \frac{1}{V_{kn} V_{k(n+1)}} &= \frac{2}{(\alpha - \beta)U_k} [L(\beta^{2k}) - 2L(\beta^{8k})] + \frac{\beta^k}{(\alpha - \beta)U_k V_k}. \end{aligned}$$

For the proof, the reader will need the following lemmas.

*Lemma 2':*

$$\begin{aligned} 2 \sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} U_{kn}} &= \frac{1}{\alpha^k U_k} + U_k \sum_{n=1}^{\infty} \frac{1}{U_{kn} U_{k(n+1)}}; \\ 2 \sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} V_{kn}} &= \frac{1}{\alpha^k V_k} + (\alpha - \beta) U_k \sum_{n=1}^{\infty} \frac{1}{V_{kn} V_{k(n+1)}}. \end{aligned}$$

Lemma 3':

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} U_{kn}} = (\alpha - \beta) [L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})];$$

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} V_{kn}} = L(\beta^{2k}) - 2L(\beta^{8k}).$$

Reference

1. A. F. Horadam. "Elliptic Functions and Lambert Series in the Summation of Reciprocals in Certain Recurrence-Generated Sequences." *Fibonacci Quarterly* 26.2 (1988):98-114.

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