LAMBERT SERIES AND THE SUMMATION OF RECIPROCALS IN CERTAIN FIBONACCI-LUCAS-TYPE SEQUENCES

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1. Introduction

Consider the sequence of real numbers defined by the recurrence relation

(1.1) $W_n = pW_{n-1} + W_{n-2}$,

where p is a strictly positive real number. Special cases of (W_n) which interest us here are:

(1.2) $U_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ (Fibonacci-type sequence),

and

(1.3) $V_n = \alpha^n + \beta^n$ (Lucas-type sequence),

where

(1.4)
$$\alpha = \frac{p + \sqrt{p^2 + 4}}{2}, \quad \beta = \frac{p - \sqrt{p^2 + 4}}{2}.$$

It is clear that

(1.5) $\alpha\beta = -1$, $\alpha > 1$, $-1 < \beta < 0$.

On the other hand, the Lambert series is defined by

(1.6) $L(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}, \quad |x| < 1.$

It has been known for a long time (see Horadam [1] for complete references) that

$$\sum_{n=1}^{\infty} \frac{1}{U_{2n}} = (\alpha - \beta)[L(\beta^2) - L(\beta^4)],$$

$$\sum_{n=1}^{\infty} \frac{1}{V_{2n-1}} = -L(\beta) + 2L(\beta^2) - L(\beta^4).$$

The purpose of this paper is to establish the following result.

Theorem 1:

(1.7) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+1}} = 2(\alpha - \beta)[L(\beta^2) - 2L(\beta^4) + 2L(\beta^8)] + \beta;$

(1.8) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{V_n V_{n+1}} = \frac{2}{\alpha - \beta}[L(\beta^2) - 2L(\beta^8)] + \frac{\beta}{(\alpha - \beta)p}.$

2. Preliminary Lemma

Lemma 1:

(2.1) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 - x^{2n+1}} = L(x) - L(x^2);$

$$(2.2)\quad \sum_{n=1}^{\infty} \frac{x^n}{1+x^n} = L(x) - 2L(x^2);$$

$$(2.3)\quad \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1+x^{2n+1}} = L(x) - 3L(x^2) + 2L(x^4).$$

(2.1) is obviously true, wherèas (2.2) follows from the identity

$$\frac{x^n}{1+x^n} = \frac{x^n}{1-x^n} - \frac{2x^{2n}}{1-x^{2n}},$$

and (2.3) follows from

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{1+x^{2n+1}} = \sum_{n=1}^{\infty} \frac{x^n}{1+x^n} - \sum_{n=1}^{\infty} \frac{x^{2n}}{1+x^{2n}}.$$

3. Proof of Theorem 1

Lemma 2:

$$(3.1)\quad 2\sum_{n=1}^{\infty} \frac{1}{\alpha^n U_n} = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+1}};$$

$$(3.2)\quad 2\sum_{n=1}^{\infty} \frac{1}{\alpha^n V_n} = \frac{1}{\alpha p} + \sum_{n=1}^{\infty} \frac{\alpha - \beta}{V_n V_{n+1}}$$

Proof: First, we have

$$\begin{aligned}\alpha U_{n+1} + U_n &= \frac{1}{\alpha - \beta}\left[\alpha\left(\alpha^{n+1} - (-1)^{n+1}\frac{1}{\alpha^{n+1}}\right) + \alpha^n - (-1)^n \frac{1}{\alpha^n}\right] \\ &= \frac{1}{\alpha - \beta}(\alpha^{n+2} + \alpha^n) = \frac{\alpha^{n+1}}{\alpha - \beta}\left(\alpha + \frac{1}{\alpha}\right) = \alpha^{n+1}.\end{aligned}$$

Thus,

$$\frac{1}{\alpha^n U_n} + \frac{1}{\alpha^{n+1} U_{n+1}} = \frac{1}{U_n U_{n+1}}.$$

By adding this term by term, we find (3.1) since $U_1 = 1$. The proof of (3.2) follows the same pattern if we observe that

$$\alpha V_{n+1} + V_n = (\alpha - \beta)\alpha^{n+1}.$$

Thus,

$$\frac{1}{\alpha^n V_n} + \frac{1}{\alpha^{n+1} V_{n+1}} = \frac{\alpha - \beta}{V_n V_{n+1}}.$$

Now, adding this term by term, we find (3.2) since $V_1 = p$.

Lemma 3:

$$(3.3)\quad \sum_{n=1}^{\infty} \frac{1}{\alpha^n U_n} = (\alpha - \beta)[L(\beta^2) - 2L(\beta^4) + 2L(\beta^8)];$$

$$(3.4)\quad \sum_{n=1}^{\infty} \frac{1}{\alpha^n V_n} = L(\beta^2) - 2L(\beta^8).$$

Proof:

$$\begin{aligned}\frac{1}{\alpha - \beta}\sum_{n=1}^{\infty} \frac{1}{\alpha^n U_n} &= \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n} - (-1)^n} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1 - (-1)^n \beta^{2n}} \\ &= \sum_{n=1}^{\infty} \frac{\beta^{4n}}{1 - \beta^{4n}} + \sum_{n=0}^{\infty} \frac{\beta^{4n+2}}{1 + \beta^{4n+2}}\end{aligned}$$

Using (1.6) with $x = \beta^4$ and (2.3) with $x = \beta^2$, we find (3.3). On the other hand, we have:

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^n V_n} = \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n} + (-1)^n} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1 + (-1)^n \beta^{2n}} = \sum_{n=1}^{\infty} \frac{\beta^{4n}}{1 + \beta^{4n}} + \sum_{n=0}^{\infty} \frac{\beta^{4n+2}}{1 - \beta^{4n+2}}.$$

Using (2.2) with $x = \beta^4$ and (2.1) with $x = \beta^2$, we find (3.4). This concludes the proof of Lemma 3. Now the proof of the theorem follows immediately from Lemmas 2 and 3.

4. Special Cases

4.1 Fibonacci-Lucas Sequences

Let $p = 1$ in (1.1) to obtain

$$W_n = W_{n-1} + W_{n-2}, \quad \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

$U_n = F_n$ is the Fibonacci sequence and $V_n = L_n$ is the Lucas sequence. Equations (1.7) and (1.8) take the following form:

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} = 2\sqrt{5}\left[L\left(\frac{3 - \sqrt{5}}{2}\right) - 2L\left(\frac{7 - 3\sqrt{5}}{2}\right) + 2L\left(\frac{47 - 21\sqrt{5}}{2}\right)\right] + \frac{1 - \sqrt{5}}{2};$$

$$\sum_{n=1}^{\infty} \frac{1}{L_n L_{n+1}} = \frac{2}{\sqrt{5}}\left[L\left(\frac{3 - \sqrt{5}}{2}\right) - 2L\left(\frac{47 - 21\sqrt{5}}{2}\right)\right] + \frac{1 - \sqrt{5}}{2\sqrt{5}}.$$

4.2 Pell and Pell-Lucas Sequences

Let $p = 2$ in (1.1) to obtain

$$W_n = 2W_{n-1} + W_{n-2}, \quad \alpha = 1 + \sqrt{2}, \quad \beta = 1 - \sqrt{2}.$$

$U_n = P_n$ is the Pell sequence, $V_n = Q_n$ is the Pell-Lucas sequence. Equations (1.7) and (1.8) take the form:

$$\sum_{n=1}^{\infty} \frac{1}{P_n P_{n+1}} = 4\sqrt{2}[L(3 - 2\sqrt{2}) - 2L(17 - 12\sqrt{2}) + 2L(577 - 408\sqrt{2})] + 1 - \sqrt{2};$$

$$\sum_{n=1}^{\infty} \frac{1}{Q_n Q_{n+1}} = \frac{1}{\sqrt{2}}[L(3 - 2\sqrt{2}) - 2L(577 - 408\sqrt{2})] + \frac{1 - \sqrt{2}}{4\sqrt{2}}.$$

5. Generalization

The following theorem generalizes the above result. It is given without proof, since the methods required exactly parallel those of Section 3. We assume that K is an odd integer.

Theorem 2:

$$\sum_{n=1}^{\infty} \frac{1}{U_{kn} U_{k(n+1)}} = \frac{2(\alpha - \beta)}{U_k}[L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})] + \frac{\beta^k}{U_k^2};$$

$$\sum_{n=1}^{\infty} \frac{1}{V_{kn} V_{k(n+1)}} = \frac{2}{(\alpha - \beta)U_k}[L(\beta^{2k}) - 2L(\beta^{8k})] + \frac{\beta^k}{(\alpha - \beta)U_k V_k}.$$

For the proof, the reader will need the following lemmas.

Lemma 2':

$$2\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} U_{kn}} = \frac{1}{\alpha^k U_k} + U_k \sum_{n=1}^{\infty} \frac{1}{U_{kn} U_{k(n+1)}};$$

$$2\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} V_{kn}} = \frac{1}{\alpha^k V_k} + (\alpha - \beta)U_k \sum_{n=1}^{\infty} \frac{1}{V_{kn} V_{k(n+1)}}.$$

Lemma 3':

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} U_{kn}} = (\alpha - \beta)[L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})];$$

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} V_{kn}} = L(\beta^{2k}) - 2L(\beta^{8k}).$$

Reference

1. A. F. Horadam. "Elliptic Functions and Lambert Series in the Summation of Reciprocals in Certain Recurrence-Generated Sequences." *Fibonacci Quarterly 26.2* (1988):98-114.

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