ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-446 *Proposed by J. A. Sjogren, University of Santa Clara, Santa Clara, CA*

Establish the following result:

Let n be a whole number and, for any rational number q, let $[q]$ be the greatest integer contained in q. Then

$$f_n = \prod_{k=1}^{\left[\frac{n-1}{2}\right]} \left(3 + 2\cos\frac{2\pi k}{n}\right).$$

Here, an *empty* product is to be interpreted as unity.

H-447 *Proposed by Albert A. Mullin, Huntsville, AL*

Determine the minimal number of one-ohm resistors necessary to realize a two-terminal circuit to within 10^{-6} ohms of e ohms of resistance. The two-terminal circuit is permitted to be non-series-parallel; i.e., we allow bridge-type sub-circuits, among others. (2) How is this minimal number of unit resistors increased if only series-parallel sub-circuits are permitted? Of course, e is the usual transcendental number.

H-448 *Proposed by T. V. Padmakumar, Trivandrum, India*

If n is any number and $a_1, a_2, \ldots, a_m$ are prime to n $(n > a_1, a_2, \ldots, a_m)$, then $(a_1a_2 \ldots a_m)^2 \equiv 1 \pmod{n}$. [The number of positive integers less than n and prime to it is denoted by $\phi(n) = m$.]

SOLUTIONS

Sum Integrals

H-425 *Proposed by Stanley Rabinowitz, Littleton, MA*
(Vol. 26, no. 4, November 1988)

Let $F_n(x)$ be the n^{th} Fibonacci polynomial $[F_1(x) = 1,\ F_2(x) = x,\ F_{n+2}(x) = xF_{n+1}(x) + F_n(x).]$

Evaluate: (a) $\displaystyle\int_0^1 F_n(x)\,dx$ (b) $\displaystyle\int_0^1 F_n^2(x)\,dx$

Solution by Paul S. Bruckman, Edmonds, WA

It may readily be shown that

(1) $$F_n(x) = \frac{a^n - b^n}{a - b}, \quad n = 1, 2, 3, \ldots,$$

where

(2) $$a \equiv a(x) \equiv \tfrac{1}{2}(x + \sqrt{x^2 + 4}), \quad b \equiv b(x) \equiv \tfrac{1}{2}(x - \sqrt{x^2 + 4}).$$

Note

(3) $$a + b = x, \quad a - b = (x^2 + 4)^{\frac{1}{2}}, \quad ab = -1.$$

We may also define the Lucas polynomials as follows:

(4) $$L_1(x) = x, \quad L_2(x) = x^2 + 2, \quad L_{n+2}(x) = xL_{n+1}(x) + L_n(x).$$

Therefore, we find that

(5) $$L_n(x) = a^n + b^n, \quad n = 1, 2, 3, \ldots .$$

It is easy to differentiate a and b (with respect to x), and we find:

(6) $$a'(x) = a/(a - b), \quad b'(x) = -b/(a - b).$$

From (6), it follows that

(7) $$L_n'(x) = nF_n(x).$$

This implies the indefinite integral:

(8) $$\int F_n(x)\,dx = \frac{1}{n}L_n(x).$$

Now $L_n(1) = L_n$, the standard Lucas numbers, while $L_n(0) = 1 + (-1)^n = 2e_n$, where e_n is the characteristic function of the even integers. Therefore, we obtain the solution to part (a):

(9) $$\int_0^1 F_n(x)\,dx = \frac{1}{n}(L_n - 2e_n).$$

Solution to part (b):

Consider the expression $S_n(x)$ defined as follows:

(10) $$S_n(x) = \sum_{k=0}^{n-1} (-1)^{n-1-k} L_{2k+1}(x)/(2k + 1), \quad n = 1, 2, 3, \ldots .$$

Differentiating S_n term by term, we obtain [using (7) and (1)]:

$$\begin{aligned} S_n'(x) &= \sum_{k=0}^{n-1} (-1)^{n-1-k} F_{2k+1}(x) = \sum_{k=0}^{n-1} (-1)^{n-1-k}\left[\frac{a^{2k+1} - b^{2k+1}}{a - b}\right] \\ &= \frac{a}{(a - b)}(a^2 + 1)^{-1}(a^{2n} - (-1)^n) - \frac{b}{(a - b)}(b^2 + 1)^{-1}(b^{2n} - (-1)^n) \\ &= (a - b)^{-2}(a^{2n} - (-1)^n + b^{2n} - (-1)^n) \\ &= (a^n - b^n)^2(a - b)^{-2}, \end{aligned}$$

or

(11) $$S_n'(x) = F_n^2(x).$$

It follows from (11) that

$$(12)\qquad \int_0^1 F_n^2(x)dx = S_n(1) - S_n(0).$$

Since $L_{2k+1}(0) = 2e_{2k+1} = 0$, we see that $S_n(0) = 0$. Hence,

$$(13)\qquad \int_0^1 F_n^2(x)dx = \sum_{k=0}^{n-1}(-1)^{n-1-k}\frac{L_{2k+1}}{2k+1}.$$

It does not appear possible to simplify the foregoing expression further, into some kind of closed form.

Also solved by O. Brugia & P. Filipponi, R. Euler, C. Georghiou, R. Andre-Jeannin, L. Kuipers, H.-J. Seiffert, J. Shallit, and D. Zeitlin. Georghiou mentioned that part (a) is identical to H-410.

Another Identity

H-426 *Proposed by Larry Taylor, Rego Park, NY*
(Vol. 26, no. 4, November 1988)

Let j, k, m, and n be integers. Prove that

$$(F_nF_{m+k-j} - F_mF_{n+k-j})(-1)^m = (F_kF_{j+n-m} - F_jF_{k+n-m})(-1)^j.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Our solution will use the following known results:

$$(1)\qquad 5F_tF_s = 5F_sF_t = L_{s+t} - (-1)^tL_{s-t}$$

and

$$(2)\qquad (-1)^tL_t = L_{-t}.$$

[For (1), see (10) and (12) on page 115 of the April 1975 issue of this journal; for (2), see exercises 3 and 9 on page 29 of *Fibonacci and Lucas Numbers* by Verner E. Hoggatt, Jr. (Boston: Houghton Mifflin, 1969.]

We now use the above information to produce the following collection of equations, each of which is equivalent to the desired result.

$$\begin{aligned}
&5(F_nF_{m+k-j} - F_mF_{n+k-j})(-1)^m\\
&= 5(F_kF_{j+n-m} - F_jF_{k+n-m})(-1)^j + [L_{n+m+k-j} - (-1)^nL_{m+k-j-n} - L_{m+n+k-j}\\
&\qquad + (-1)^mL_{n+k-j-m}](-1)^m\\
&= [L_{n+k+j-m} - (-1)^kL_{j+n-m-k} - L_{n+j+k-m} + (-1)^jL_{k+n-m-j}](-1)^j\\
&\qquad - (-1)^{m+n}L_{m+k-j-n} + L_{n+k-j-m}\\
&= -(-1)^{k+j}L_{j+n-m-k} + L_{k+n-m-j} + (-1)^{m+n-k-j}L_{m+k-j-n}\\
&= L_{-(m+k-j-n)} + (-1)^{2k}(-1)^{-2n}(-1)^{m+n-k-j}L_{m+k-j-n}\\
&= L_{-(m+k-j-n)} + (-1)^{m+k-j-n}L_{m+k-j-n}\\
&= L_{-(m+k-j-n)}
\end{aligned}$$

Since the last equality holds by (2), our solution is complete.

Also solved by P. Bruckman, P. Filipponi, C. Georghiou, R. Hendel, R. Andre-Jeannin, L. Kuipers, H.-J. Seiffert, S. Singh, and the proposer.

A Recurrent Composition

H-427 *Proposed by Piero Filipponi, Rome, Italy (Vol. 26, no. 4, November 1988)*

Let $C(n, k) = C_1(n, k)$ denote the binomial coefficient $\binom{n}{k}$.

Let $C_2(n, k) = C[C(n, k), k]$ and, in general,

$C_i(n, k) = C(C\{\ldots[C(n, k), k]\})$.

For given n and i, is it possible to determine the value k_0 of k for which

$C_i(n, k_0) > C_i(n, k) \quad (k = 0, 1, \ldots, n;\ k \neq k_0)$?

Solution by Paul S. Bruckman, Edmonds, WA

We may make the problem a bit more precise by redefining k_0 *uniquely* as follows:

(1) $C_i(n, k_0) \geq C_i(n, k), \quad k = 0, 1, 2, \ldots, n,$

with k_0 being the *smallest* integer with this property.

Of course, $k_0 = k_0(n, i)$, dependent on the values of n and i.

We may readily show that k_0 as thus defined is uniquely determined. We see that, for all $n > 1$,

(2) $C_i(n, 0) = 1; \quad C_i(n, 1) = C_i(n, n-1) = n; \quad C_i(n, n) = 0,$

from which the conclusion follows.

The construction of k_0 is much more difficult in the general case; however, for $i = 1$, the solution is well known, namely:

(3) $k_0(n, 1) = [n/2]$.

In other words, given n, the maximum value of $\binom{n}{k}$ is assumed at $k = [n/2]$ (and also at $k = [\frac{1}{2}(n + 1)]$, which we ignore, due to our uniqueness definition).

Even the case $i = 2$ readily becomes formidable; however, we may make some statistical inferences, by means of Stirling's formula, which may have some validity as $n \to \infty$. By means of a TI-60 Scientific Calculator, the following table was obtained:

n	$k_0(n, 2)$	n	$k_0(n, 2)$	n	$k_0(n, 2)$
0	0	9	6	18	12
1	0	10	6	19	13
2	1	11	7	20	13
3	1	12	8	21	14
4	2	13	8	22	15
5	3	14	9	23	15
6	4	15	10	24	16
7	4	16	10	25	17
8	5	17	11	26	17
				27	18

It appears from the table that $k_0(n, 2) \sim 2n/3$, at least asymptotically. We conjecture that, given i, a constant $\theta(i)$ exists such that

(4) $\quad k_0(n, i) \sim n \cdot \theta(i)$, as $n \to \infty$.

To test this hypothesis, we note that, for most values of k, $C_i(n, k)$ is of a much higher order of magnitude than k. Using the approximate relation

(5) $\quad \binom{n}{k} \sim \frac{n^k}{k!}$, valid when n/k is large,

as well as the obvious recursion

(6) $\quad C_{i+1}(n, k) = \binom{C_i(n, k)}{k}$,

we obtain the asymptotic relation

(7) $\quad C_{i+1}(n, k) \sim \dfrac{(C_i(n, k))^k}{k!}$, for all except the extreme values of k.

We may make another observation, namely, that the sequence

$$K_n = \left(k_0(n, i)\right)_{i=1}^{\infty}$$

is nondecreasing for a given n. To see this, let $u = k_0(n, i)$, $v = k_0(n, i+1)$. Then, by definition, $C_i(n, v) \le C_i(n, u)$ and $C_{i+1}(n, u) \le C_{i+1}(n, v)$. Thus,

$$\binom{C_i(n, v)}{u} \le \binom{C_i(n, u)}{u} \le \binom{C_i(n, v)}{v},$$

which implies

(8) $\quad u \le v$.

We assume the relationship in (4); letting $\theta = \theta(2)$ and applying Stirling's formula, we obtain:

(9) $\quad C_1(n, \theta n) \sim A_1 n^{-\frac{1}{2}} B_1^n$, where $A_1 = (2\pi\theta(1-\theta))^{-\frac{1}{2}}$, $B_1 = (\theta^\theta(1-\theta)^{1-\theta})^{-1}$.

Since $C_1(n, \theta n)$ is generally much larger than θn, we may apply (5) with k set to equal θn and obtain, after a second application of Stirling's formula:

(10) $\quad C_2(n, \theta n) \sim A_2 n^{-\frac{1}{2}-\frac{3}{2}n\theta} B_2^n C_2^{n^2}$, where $A_2 = (2\pi\theta)^{-\frac{1}{2}}$,

$$B_2 = (eA_1/\theta)^\theta, \text{ and } C_2 = B_1^\theta.$$

Note that $C_2(n, \theta n)$ is dominated by the term $C_2^{n^2} = (B_1^\theta)^{n^2}$. We may perform a similar computation, only this time letting $\theta = \theta(3)$; we then find that

$$C_3(n, \theta n) \sim A_1 n^{-\frac{1}{2}(1 + 3n\theta + n^2\theta^2)} B_3^n C_3^{n^2} D_3^{n^3},$$

where B_3 and C_3 are constants and $D_3 = B_1^{\theta^2}$ [note that A_1 and B_1 are defined as in (9), but have different values, since a different value of θ is used to compute them]. We note that $C_3(n, \theta n)$ is dominated by the term

$$D_3^{n^3} = (B_1^{\theta^2})^{n^3}.$$

It appears that if we can determine θ such that B_1^θ and $B_1^{\theta^2}$ are maximized, we can determine $k_0(n, 2)$ and $k_0(n, 3)$, respectively. Generalizing further, we see that $\theta(i) = \theta$ may be determined, according to this line of reasoning, by maximizing the expression

(11) $\quad B_1^{\theta^{i-1}} = \theta^{-\theta^i}(1-\theta)^{-(1-\theta)\theta^{i-1}}$.

Since $\theta \in (0, 1)$, we see that the above expression exceeds unity; hence, its logarithm is positive. Letting $f(\theta, i)$ equal the logarithm, we thus seek to maximize the expression:

(12) $\quad f(\theta, i) = -\theta^i \log \theta - \theta^{i-1}(1-\theta) \log (1-\theta)$.

Note that $f > 0$ if $\theta \in (0, 1)$, while

$$f(0, i) = \lim_{\theta \to 0^+} f(\theta, i) = 0 \quad \text{and} \quad f(1, i) = \lim_{\theta \to 1^-} f(\theta, i) = 0;$$

hence, f does indeed assume a maximum value for some $\theta \in (0, 1)$. To find this value of $\theta(i)$, we need to solve the equation $f'(\theta, i) = 0$, which yields the transcendental equation

$$\left[\theta - \left(\frac{i-1}{i}\right)\right] \log(1-\theta) - \theta \log \theta = 0. \tag{13}$$

Clearly, for $i = 1$, we obtain the value $\theta = \theta(1) = \frac{1}{2}$, which is correct. For $i = 2$, we obtain, after some computation on the TI-60, the value

$$\theta(2) \doteq .7035060764. \tag{14}$$

For a few other values of i, the following values were obtained, as the solutions of (13):

$$\begin{aligned} &\theta(3) \doteq .78783\,98702, \quad \theta(4) \doteq .83417\,45130, \\ &\theta(5) \doteq .86358\,23417, \quad \theta(6) \doteq .88395\,71002. \end{aligned} \tag{15}$$

It is conjectured that, asymptotically at least, the above algorithm yields the appropriate value of $\theta(i)$, such that $k_0(n, i)$ is validly obtained in (4). As for the smaller values of n, no inference is provided by this procedure.

Same Difference

H-428 *Proposed by Larry Taylor, Rego Park, NY (Vol. 27, no. 1, February 1989)*

Let j, m, and n be integers. Let a and b be relatively prime even-odd integers with b not divisible by 5. Let $A_n = aL_n + bF_n$. Then $A_n = A_{n+1} - A_{n-1}$ with initial values $A_1 = b + a$, $A_{-1} = b - a$.

Prove that the following three numbers

$$(2F_{n-j}A_{m-j}, F_{n+j}A_{m+j}, 2F_{2j}A_{n+m})$$

are in arithmetic progression.

Solution by Russell Jay Hendel, Dowling College, Oakdale, NY

We solve the problem without the divisibility restrictions on a and b. First, observe that $[2x, y, 2z]$ are in arithmetic progression iff $x + z = y$. Next, observe that A_n are linear combinations of the F_n and L_n. Therefore, it suffices to prove the two equations,

$$F_{n-j}L_{m-j} + F_{2j}L_{n+m} = F_{n+j}L_{m+j} \quad \text{and} \quad F_{n-j}F_{m-j} + F_{2j}F_{n+m} = F_{n+j}F_{m+j}.$$

We will prove only second equation, proof of the first equation being similar.

By Binet's formula (which clearly holds also for subscripts with negative values), we reduce proof of this equation to proof of the equality

$$\begin{aligned} &[p^{n+m-2j} + q^{n+m-2j}] - (p^{n+m}q^{2j} + p^{2j}q^{n+m}) \\ &= [q^{n-j}p^{m-j} + p^{n-j}q^{m-j}] - (p^{n+j}q^{m+j} + p^{m+j}q^{n+j}), \end{aligned}$$

with p and q the roots of $x^2 - x - 1 = 0$. To complete the proof, we multiply the bracketed expression on each side of this equality by $1 = (pq)^{2j}$. This shows that both sides of the equation are zero and completes the proof.

Also solved by P. Bruckman, P. Filipponi, R. Andre-Jeannin, L. Kuipers, Y. H. Harris Kwong, and the proposer.

Editorial Note:

The editor welcomes solutions of *any* previously proposed problem. Also, in order to avoid misreading proposals or solutions, it would be appreciated if submitted material is typed or printed.

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(Continued from page 354)

Once in our medium-sized auditorium, we were intrigued (and assisted) by "the wonders technology had wrought": there were two overhead projectors, and blackboards—ugh, whiteboards!—came from everywhere; up and down they went, above and below, over and across, sometimes interceded by a screen that appeared from nowhere..., and all of it happened by the touch of a button, skillfully activated by the cognoscenti.

Of course, there was not only food for the mind and the soul, but also for the stomach. Wake Forest University graciously treated us to daily morning and afternoon coffee breaks, and the president, Dr. Thomas K. Hearn, Jr., hosted a wine and cheese reception on campus.

Even though our daily meetings took place from 9:00 a.m. till noon, and from 2:00 p.m. to 5:00 p.m., we did not ALWAYS work. In midweek, the afternoon was freed, and we took off to Doughton Park in the beautiful Blue Ridge Mountains of North Carolina. There the group dispersed to enjoy the magnificent scenery with a choice of several hiking trails that offer spectacular vistas. Those of us who preferred less energetic activities, relaxed at a coffee shop where we did, what we seem to do best, or at least most often, and with pleasure: exchange mathematical ideas. All this was followed by a lavish, typically North Carolinian dinner at Shatley Springs.

The next day we celebrated our customary evening banquet. It was held on campus, and was at once elegant and friendly, somehow reflecting the spirit of our group. We speak with many different foreign accents, yet we all understand each other, professionally and personally. The magnetism of our beloved discipline has somehow promoted a very special bond of friendship. Many of us had been together at some of the past conferences. Quite a few papers exhibited the resulting kindling of common mathematical interests which culminated in joint authorships.

Maybe several of you are already gathering your thoughts for our next Conference. "*Auf Wiedersehen*," then, in 1992 at St. Andrews University, Scotland.

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