ASYMPTOTIC POSITIVENESS OF LINEAR RECURRENCE SEQUENCES

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Dedicated to Professor L. Kuipers
on the occasion of his 80th birthday

Suppose that the first several terms of a sequence are given, then it is not so easy to predict the asymptotic behavior of this sequence. But once we know that this given sequence is a linear recurrence sequence, we can determine the asymptotic behavior through its recurrence formula.

Indeed, John R. Burke and William A. Webb [1] considered real linear recurrence sequences $\{u_n\}_{n=0}^{\infty}$ of order d defined by

(1) $$u_{n+d} = a_{d-1}u_{n+d-1} + a_{d-2}u_{n+d-2} + \cdots + a_0u_n \text{ for } n \geq 0,$$

where $a_0, a_1, \ldots, a_{d-1}$ are real numbers, with its corresponding characteristic equation:

(2) $$p(x) = x^d - a_{d-1}x^{d-1} - \cdots - a_1x - a_0 = 0.$$

They obtained a criterion for the asymptotic positiveness of linear recurrence sequences (1) if the corresponding characteristic equation has distinct roots. Here we call a sequence $\{u_n\}_{n=0}^{\infty}$ *asymptotically positive* if there exists a natural number n_0 such that

$$u_n > 0 \text{ for all } n \geq n_0.$$

In particular, if the above n_0 is equal to zero, we call this sequence $\{u_n\}_{n=0}^{\infty}$ *totally positive.*

In this note, we shall give a criterion of asymptotic positiveness of real linear recurrence sequences $\{u_n\}_{n=0}^{\infty}$ (1) of order d, when their characteristic equations have multiple roots.

Let us recall a general representation formula for u_n. We assume that the corresponding characteristic equation (2) of $\{u_n\}_{n=0}^{\infty}$ has roots $\lambda_1, \lambda_2, \ldots, \lambda_p$ with corresponding multiplicities $m_1, m_2, \ldots, m_p$. Then there exist polynomials $b_1, b_2, \ldots, b_p$ with degree $b_i \leq m_i - 1$ for $i = 1, 2, \ldots, p$, where the coefficients of polynomials $b_1, b_2, \ldots, b_p$ depend only on the roots of the characteristic equation (2) and the initial values of this recurrence sequence. Then, we have, for all $n \geq 0$,

(3) $$u_n = b_1(n)\lambda_1^n + b_2(n)\lambda_2^n + \cdots + b_p(n)\lambda_p^n.$$

The detailed discussion of this representation (3) can be found, for example, in Władysław Narkiewicz [4] or Alecksei I. Markuševič [2].

Without loss of generality, we arrange the roots $\lambda_1, \lambda_2, \ldots, \lambda_p$ according to their moduli as

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_p|.$$

Suppose first that λ_2 is the complex conjugate of λ_1, λ_1 is not real, and

(4) $$|\lambda_1| = |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_p|.$$

We assume further that the sum of the first two terms of (3), denoted by

(5) $$v_n = b_1(n)\lambda_1^n + b_2(n)\lambda_2^n,$$

does not vanish for infinitely many n. Then

(6) $\qquad u_n = v_n + o(v_n)$

holds for all sufficiently large n (see Nagasaka, Kanemitsu, & Shiue [3]).

Since $\{u_n\}_{n=0}^{\infty}$ is a real sequence, we get

$$b_2(n) = \overline{b_1(n)}$$

and

$$\begin{aligned} v_n = b_1(n)\lambda_1^n + b_2(n)\lambda_2^n &= b_1(n)(re^{2\pi i\theta})^n + \overline{b_1(n)}(re^{-2\pi i\theta})^n \\ &= b_1(n)r^n e^{2\pi in\theta} + (\overline{b_1(n)r^n e^{2\pi in\theta}}) \\ &= 2\ \mathrm{Re}\{b_1(n)r^n e^{2\pi in\theta}\}, \end{aligned}$$

where $\lambda_1 = re^{2\pi i\theta}$ and θ is not a multiple of π (since λ_1 is not real). Now, if we write

$$b_1(n) = c_k n^k + c_{k-1}n^{k-1} + \cdots + c_0,$$

where $c_0, c_1, \ldots, c_k$ are complex numbers determined by the roots $\lambda_1, \lambda_2, \ldots, \lambda_p$ and initial values $u_0, u_1, \ldots, u_{d-1}$ with nonzero c_k, $k \le m_1 - 1$. Then

$$\begin{aligned} v_n &= 2\ \mathrm{Re}(c_k n^k r^n e^{2\pi in\theta}) + o(n^k r^n) \\ &= 2n^k r^n\ \mathrm{Re}(c_k)\cos(2\pi n\theta) + o(n^k r^n) \text{ for large } n. \end{aligned}$$

Since θ is not a multiple of π, v_n takes negative values for infinitely many n, by applying the same argument as in the proof of Theorem 1 in Burke & Webb [1]. Hence, by (6), the original linear recurrence sequence $\{u_n\}$ is not asymptotically positive for this case. Summarizing the above discussion, we have

Theorem 1: Suppose that the roots $\lambda_1, \lambda_2, \ldots, \lambda_p$ of the characteristic equation of $\{u_n\}_{n=0}^{\infty}$ satisfy (4) and that λ_1 and λ_2 are complex conjugates of each other and are not real. Assume that v_n does not vanish for infinitely many n, then the linear recurrence sequence $\{u_n\}_{n=0}^{\infty}$ is not asymptotically positive.

Secondly, we assume again the relation (4) with real λ_1 and λ_2, that is, $-\lambda_2 = \lambda_1$. We denote the leading coefficients of the polynomials $b_1(n) + b_2(n)$ and $b_1(n) - b_2(n)$ by A and B, respectively, and assume further that $AB \ne 0$ for all sufficiently large n. Say that $b_1(n) + b_2(n)$ has degree k, $b_1(n) - b_2(n)$ has degree ℓ. Then (8) holds for all sufficiently large n.

Hence, we have that, for all sufficiently large even n,

(7) $\qquad u_n = An^k\lambda_1^n + o(n^k\lambda_1^n)$

and, for all sufficiently large odd n, we get

(8) $\qquad u_n = Bn^\ell\lambda_1^n + o(n^\ell\lambda_1^n)$.

Thus, we obtain

Theorem 2: Suppose that the roots $\lambda_1, \lambda_2, \ldots, \lambda_p$ of the characteristic equation of $\{u_n\}_{n=0}^{\infty}$ satisfy (4) and $0 < \lambda_1 = -\lambda_2$ that are real. Assume further that the leading coefficients A and B of the polynomials $b_1(n) + b_2(n)$ and $b_1(n) - b_2(n)$ are positive. Then $\{u_n\}_{n=0}^{\infty}$ is asymptotically positive.

We now leave assumption (4). Then, we have either

(9) $\qquad |\lambda_1| = |\lambda_2| = |\lambda_3| = \cdots = |\lambda_j| > |\lambda_{j+1}| \ge \cdots \ge |\lambda_p|,$

for some $j > 2$, or

(10) $\qquad |\lambda_1| > |\lambda_2| \ge \cdots \ge |\lambda_p|.$

First, let us consider the case (10). From the fact that the coefficients of the characteristic equation $a_0, a_1, \ldots, a_{d-1}$ are all real, λ_1 must be real.

Also, if $b_1(n)$ is not identically zero, we get

(11) $\qquad u_n = Cn^m\lambda_1^n + o(n^m\lambda_1^n),$

where C is the leading coefficient of the polynomial $b_1(n)$ of degree $m \le m_1 - 1$. Thus, we obtain

Theorem 3: Suppose that the roots $\lambda_1, \lambda_2, \ldots, \lambda_p$ of the characteristic equation of $\{u_n\}_{n=0}^\infty$ satisfy (10). Assume further that the polynomial $b_1(n)$ is not identically zero, that λ_1 is positive, and that the leading coefficient C of $b_1(n)$ is also positive. Then the linear recurrence sequence $\{u_n\}_{n=0}^\infty$ is asymptotically positive.

For the remaining case (9), we need to divide into the following three subcases:

(i) j is even, all λ_ℓ are not real for $\ell = 1, 2, \ldots, j$ and λ_{2i} is the complex conjugate of λ_{2i-1} for $i = 1, 2, \ldots, j/2$. We assume further that $b_1(n)$, $b_2(n), \ldots, b_p(n)$ do not vanish for all $n \ge n_0$.

Then, applying Theorem 1, $\{u_n\}_{n=0}^\infty$ is not asymptotically positive.

(ii) j is even, $0 < \lambda_1 = -\lambda_2$ are real, all other λ_ℓ for $\ell = 3, 4, \ldots, j$ are not real, and λ_{2i} is the complex conjugate of λ_{2i-1} for $i = 2, 3, \ldots, j/2$. We suppose again that $b_1(n), b_2(n), \ldots, b_j(n)$ do not vanish for all $n \ge n_0$.

Then $\{u_n\}_{n=0}^\infty$ is asymptotically positive if the leading coefficients A, B of $b_1(n) + b_2(n)$ and $b_1(n) - b_2(n)$, respectively, are both positive for all sufficiently large n and either

$$\min\{\deg(b_1(n) + b_2(n)), \deg(b_1(n) - b_2(n))\} \text{ is greater than}$$

$$\max_{i=2, 3, \ldots, j/2}\{\deg(2\,\mathrm{Re}(b_{2i-1}(n)))\}$$

or

$\min(A, B) - 1$ is greater than all the leading coefficients of $2\,\mathrm{Re}(b_{2i-1}(n))$ for which

$$\min\{\deg(b_1(n) + b_2(n)), \deg(b_1(n) - b_2(n))\}$$
$$= \deg\{2\,\mathrm{Re}(b_{2i-1}(n))\} \text{ for } i = 2, 3, \ldots, j/2.$$

(iii) j is odd, $0 < \lambda_1$ is real, all other λ_ℓ are not real for $\ell = 2, 3, \ldots, j$ and λ_{2i+1} is the complex conjugate of λ_{2i} for $i = 1, 2, \ldots, [j/2]$.

Then $\{u_n\}_{n=0}^\infty$ is asymptotically positive if the leading coefficient C of $b_1(n)$ is positive and either $\deg(b_1(n))$ is greater than

$$\max_{i=1, 2, \ldots, [j/2]}\{\deg(2\,\mathrm{Re}(b_{2i}(n)))\}$$

or $C - 1$ is greater than all the leading coefficients of $2\,\mathrm{Re}(b_{2i}(n))$ for which

$$\deg(b_1(n)) = \deg(2\,\mathrm{Re}(b_{2i}(n))) \text{ for } i = 1, 2, \ldots, [j/2].$$

We assume always the nonvanishing property of all $b_\ell(n)$ for $\ell = 1, 2, \ldots, j$, for the case (9). If some of the $b_\ell(n)$ are identically zero, say $b_k(n)$, then we simply ignore these terms $b_k(n)\lambda_k^n$, and it is sufficient to trace the above discussion.

Finally, we give explicit conditions for a real linear recurrence sequence of order 2 or of order 3 to be asymptotically positive.

We denote $\{s_n\}_{n=0}^\infty$ a linear recurrence sequence of order 2 with recurrence formula $s_{n+2} = a_1s_{n+1} + a_0s_n$. First, we assume that its corresponding characteristic equation of degree 2 has only one real double root $\alpha \ne 0$. Then, $a_1 = 2\alpha$ and $a_0 = -\alpha^2$ and the n^{th} term s_n can be represented by

$$s_n = (p_1 n + p_2)\alpha^n \text{ for } n \geq 0.$$

By solving the system of equations

$$\begin{cases} s_0 = p_2 \\ s_1 = (p_1 + p_2)\alpha, \end{cases}$$

we obtain

$$p_1 = (s_1 - s_0\alpha)/\alpha.$$

Applying the discussion of Theorem 3 above, we have

Theorem 4: Suppose the characteristic equation of a linear recurrence sequence $\{s_n\}_{n=0}^{\infty}$ has only one real double nonzero root α. Sequence $\{s_n\}_{n=0}^{\infty}$ is asymptotically positive if and only if $\alpha > 0$ and either $s_1 > s_0\alpha$ or $s_0 > 0$ and $s_1 = s_0\alpha$.

Corollary 4.1: Under the same assumption as in Theorem 4, the sequence $\{s_n\}_{n=0}^{\infty}$ is asymptotically positive if and only if $a_1 > 0$ and either $2s_1 > a_1 s_0$ or $s_0 > 0$ and $2s_1 = a_1 s_0$.

By using the relation between α and the a_i's, this Corollary follows immediately from Theorem 4.

Let us recall the case where the characteristic equation of a linear recurrence sequence $\{s_n\}_{n=0}^{\infty}$, that is,

$$\lambda^2 - a_1\lambda - a_0 = 0, \tag{12}$$

has two distinct roots.

Theorem 5: Let $D = a_1^2 + 4a_0$ be the discriminant of equation (12) of degree 2. Suppose the characteristic equation of $\{s_n\}_{n=0}^{\infty}$ has two distinct roots α_1 and α_2. This sequence $\{s_n\}_{n=0}^{\infty}$ is asymptotically positive if and only if $\sqrt{D}$ is real and one of the following four conditions is satisfied:

(i) $a_1 = 0,\ s_0 > 0,\ s_1 > 0.$

(ii) $a_1 > 0,\ 2s_1 > (a_1 - \sqrt{D})s_0.$

(iii) $a_1 > 0,\ 2s_1 = (a_1 - \sqrt{D})s_0,\ s_0 > 0,\ a_0 < 0.$

(iv) $a_1 < 0,\ 2s_1 = (a_1 + \sqrt{D})s_0,\ s_0 > 0,\ a_0 > 0.$

Proof: Suppose first that $\sqrt{D}$ is purely imaginary. Then α_2 is the complex conjugate of α_1 and the n^{th} term s_n can be represented by

$$s_n = c_1\alpha_1^n + \bar{c}_1\bar{\alpha}_1^n,$$

since $\{s_n\}_{n=0}^{\infty}$ is a sequence of real numbers. We now apply Theorem 1.

For $\{s_n\}_{n=0}^{\infty}$, v_n, as defined by (5), is identical to s_n. The nonvanishing assumption of $s_n = v_n$ is naturally satisfied, since otherwise $\{s_n\}_{n=0}^{\infty}$ becomes the sequence of 0's which is not asymptotically positive. Hence, all assumptions of Theorem 1 are fulfilled. Thus, for purely imaginary $\sqrt{D}$, $\{s_n\}_{n=0}^{\infty}$ is not asymptotically positive by Theorem 1.

Now we get necessarily that if $\sqrt{D}$ is positive real then $\alpha_1 > \alpha_2$. Condition (i) is already treated in the proof of Theorem 3 [1]. For the remaining cases, (ii), (iii), and (iv), we use a representation formula of s_n,

$$s_n = c_1\alpha_1^n + c_2\alpha_2^n,$$

with

$$c_1 = \frac{s_1 - s_0\alpha_2}{\alpha_1 - \alpha_2}, \quad c_2 = \frac{s_0\alpha_1 - s_1}{\alpha_1 - \alpha_2}.$$

In addition to case (ii) treated already in Theorem 3 [1], we are forced to add condition (iii), since c_1 may be zero. If $c_1 = 0$ with positive a_1, then

$$s_n = \frac{s_0\alpha_1 - s_1}{\alpha_1 - \alpha_2}\alpha_2^n.$$

Thus, we require that $s_0\alpha_1 - s_1 > 0$ and $\alpha_2 > 0$, from which we deduce $u_0 > 0$ and $a_0 < 0$.

If $a_1 < 0$ with real positive $\sqrt{D}$, then $\alpha_2 < 0$ and $|\alpha_1| < |\alpha_2|$. For asymptotic positiveness of $\{s_n\}_{n=0}^{\infty}$, we require that $c_2 = 0$, $c_1 > 0$, and $\alpha_1 > 0$. Rewriting these three conditions, we obtain (iv).

The sufficiency part of Theorem 5 is almost immediate from the representation formula of s_n. Q.E.D.

Remark: Combining Theorems 4 and 5, we obtain a complete characterization for asymptotic positiveness of linear recurrence sequences $\{s_n\}_{n=0}^{\infty}$ of order 2 in terms only of the coefficients of the recurrence formula and of the initial values.

Now we consider a linear recurrence sequence $\{t_n\}_{n=0}^{\infty}$ of order 3 with recurrence relation

$$t_{n+3} = a_2t_{n+2} + a_1t_{n+1} + a_0t_n.$$

Burke & Webb [1] give a sufficient condition for $\{t_n\}_{n=0}^{\infty}$ to be asymptotically positive.

Theorem 6: Suppose the characteristic equation of $\{t_n\}_{n=0}^{\infty}$ has distinct roots $\alpha_1, \alpha_2, \alpha_3$ and that they satisfy either

(13) $\qquad |\alpha_1| > |\alpha_2| \geq |\alpha_3|$

or

$$|\alpha_1| = |\alpha_2| > |\alpha_3| \text{ and } \alpha_2 \text{ is the complex conjugate of } \alpha_1.$$

If $\alpha_1 > 0$ and $c_1 > 0$, then $\{t_n\}_{n=0}^{\infty}$ is asymptotically positive where t_n is written as

(14) $\qquad t_n = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n.$

Keeping the assumption of distinct roots, Theorem 6 does not cover the following cases:

(i) $\alpha_1 = -\alpha_2$ with real α_1.

(ii) α_2 is the complex conjugate of α_1 and the roots satisfy

$$|\alpha_1| = |\alpha_2| = |\alpha_3|.$$

Case (i) can be treated using Theorem 2; however, (ii) is a special case of (9) which brings certain difficulty to determine $\{t_n\}_{n=0}^{\infty}$ to be asymptotically positive.

Burke & Webb give another elegant sufficient condition for $\{t_n\}_{n=0}^{\infty}$ to be asymptotically positive as Theorem 2 in [1], but they implicitly assume (13) and also that $c_1 \neq 0$ in (14). In order to obtain the necessary and sufficient conditions for $\{t_n\}_{n=0}^{\infty}$ to be asymptotically positive as in Theorem 5 with the assumption of distinct roots, there are too many cases split according to the vanishingness of the coefficients in (14). We can treat all of these cases; however, we shall give necessary and sufficient conditions for $\{t_n\}_{n=0}^{\infty}$ to be asymptotically positive only when the characteristic equation has multiple roots, since originally we planned to generalize the results of Burke & Webb [1] for multiple roots.

Thus, we assume that the characteristic equation of $\{t_n\}_{n=0}^{\infty}$ of order 3 has multiple roots. In order to determine conditions for $\{t_n\}_{n=0}^{\infty}$ to be asymptotically positive, Theorem 3 assumes that it is sufficient to consider only the following two cases:

(I) The corresponding characteristic equation of degree 3 has only one triple real root β.

(II) The corresponding characteristic equation of degree 3 has one double real root $\beta \neq 0$ and another real root γ with $|\beta| \geq |\gamma|$.

Let us treat case (I). The n^{th} term t_n is represented by

$$t_n = (q_1 n^2 + q_2 n + q_3)\beta^n \text{ for } n \geq 0.$$

Solving the system of equations

$$\begin{cases} t_0 = q_3 \\ t_1 = (q_1 + q_2 + q_3)\beta \\ t_2 = (4q_1 + 2q_2 + q_3)\beta^2, \end{cases}$$

we get

$$q_1 = \frac{t_2 - 2t_1 \ \ + t_0\beta^2}{2\beta^2}, \quad q_2 = \frac{-t_2 + 4t_1\beta - 3t_0\beta^2}{2\beta^2}, \quad q_3 = t_0.$$

Thus, in case (I), the sequence $\{t_n\}_{n=0}^{\infty}$ is asymptotically positive if and only if $\beta > 0$ and either

(15) $\quad t_2 - 2t_1\beta + t_0\beta^2 > 0$

or

(16) $\quad t_2 - 2t_1\beta + t_0\beta^2 = 0 \quad \text{and} \quad -t_2 + 4t_1\beta - 3t_0\beta^2 > 0$

or

(17) $\quad t_2 - 2t_1\beta + t_0\beta^2 = -t_2 + 4t_1\beta - 3t_0\beta^2 = 0 \quad \text{and} \quad t_0 > 0.$

Condition (16) can be reduced to

(18) $\quad t_1 > t_0\beta \quad \text{and} \quad t_2 = 2t_1\beta - t_0\beta^2.$

Condition (17) can also be reduced to

(19) $\quad t_2 = t_0\beta^2, \quad t_1 = t_0\beta, \quad \text{and} \quad t_0 > 0.$

Summarizing the above argument, we have

Theorem 7: Let $\{t_n\}_{n=0}^{\infty}$ be a linear recurrence sequence of order 3. Suppose the characteristic equation of $\{t_n\}_{n=0}^{\infty}$ has only one triple real root β. The sequence $\{t_n\}_{n=0}^{\infty}$ is asymptotically positive if and only if $a_0 > 0$, $a_2 > 0$, and one of the following three conditions holds:

(i) $3t_2 - 2a_2t_1 - a_1t_0 > 0$.

(ii) $3t_2 - 2a_2t_1 - a_1t_0 = 0$ and $3t_2 - 4a_2t_1 - 3a_1t_0 < 0$.

(iii) $3t_2 - 2a_2t_1 - a_1t_0 = 3t_2 - 4a_2t_1 - 3a_1t_0 = 0$ and $t_0 > 0$.

These three conditions are mentioned in (15), (18), and (19) above. We need only rewrite them as the relations

$$a_2 = 3\beta, \quad a_1 = -3\beta^2, \quad a_0 = \beta^3,$$

since β is the triple multiple root of the characteristic equation

$$\lambda^3 - a_2\lambda^2 - a_1\lambda - a_0 = 0. \quad \text{Q.E.D.}$$

For case (II), the n^{th} term t_n is represented by

$$t_n = (q_1 n + q_2)\beta^n + h\gamma^n.$$

Thus, we have

$$h = \frac{(\beta^2 + 2\gamma^2)t_0 - 2\beta t_1 + t_2}{(\beta - \gamma)^2}, \quad q_1 = \frac{\gamma(\beta + 2\gamma)t_0 - (\beta + \gamma)t_1 + t_2}{\beta(\beta - \gamma)},$$

and

$$q_2 = \frac{-\gamma(2\beta + \gamma)t_0 + 2\beta t_1 - t_2}{(\beta - \gamma)^2}.$$

We now divide into two subcases:

(IIa) $|\beta| > |\gamma|$.

In this case, the sequence $\{t_n\}_{n=0}^{\infty}$ is asymptotically positive if and only if $\beta > 0$ and either $q_1 > 0$ or $q_1 = 0$ and $q_2 > 0$ or $q_1 = q_2 = 0$, $h > 0$, and $\gamma > 0$.

(IIb) $|\beta| = |\gamma|$.

In this case, the sequence $\{t_n\}_{n=0}^{\infty}$ is asymptotically positive if and only if either $\beta > 0$ and $q_1 > 0$ or $\beta > 0$, $\gamma > 0$, $q_1 = 0$, $q_2 + h > 0$, and $q_2 > h$ or $\beta < 0$, $q_1 = 0$, $q_2 + h > 0$, and $q_2 < h$ or $q_1 = q_2 = 0$, $h > 0$, and $r > 0$.

Remark: For an arbitrary given linear recurrence sequence $\{t_n\}_{n=0}^{\infty}$, we can give explicit conditions for $\{t_n\}_{n=0}^{\infty}$ to be asymptotically positive when the characteristic equation has one real double root β and another real root γ with γ in terms of only the coefficients of the recurrence formula and of the initial values as in Theorem 6, since we have $a_2 = 2\beta + \gamma$, $a_1 = -2\beta\gamma - \beta^2$, and $a_0 = \beta^2\gamma$.

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