Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-449 Proposed by Ioan Sadoveanu, Ellensburg, WA

Let $G(x) = x^k + a_1 x^{k-1} + \cdots + a_k$ be a polynomial with c as a root of order p. If $G^{(p)}(x)$ denotes the p^{th} derivative of G(x), show that

 $\left\{\frac{n^{p}c^{n-p}}{G^{(p)}(c)}\right\} \text{ is a solution to the recurrence}$ $u_{n} = c^{n-k} - a_{1}u_{n-1} - a_{2}u_{n-2} - \dots - a_{k}u_{n-k}.$

H-450 Proposed by R. André-Jeannin, Sfax, Tunisia

Compare the numbers

$$\Theta = \sum_{n=1}^{\infty} \frac{1}{F_n}$$

and

$$\Theta' = 2 + \sum_{n=1}^{\infty} \frac{1}{F_n (2F_{n-1}^2 + (-1)^{n-1}) (2F_n^2 + (-1)^n)}.$$

H-451 Proposed by T. V. Padmakumar, Trivandrum, South India

If p is a prime and x and a are positive integers, show

$$\binom{x+ap}{p} - \binom{x}{p} \equiv a \pmod{p}.$$

SOLUTIONS

Pell Mell

<u>H-424</u> Proposed by Piero Filipponi & Adina Di Porto, Rome, Italy (Vol. 26, no. 3, August 1988)

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Let F_n and P_n denote the Fibonacci and Pell numbers, respectively. Prove that, if F_p is a prime (p > 3), then either $F_p | P_H$ or $F_p | P_{H+1}$, where $H = (F_p - 1)/2.$

Solution by Paul S. Bruckman, Edmonds, WA

Let $q = F_p > 3$, a prime. Since $p \equiv \pm 1 \pmod{6}$, it is clear from a table of congruences (mod 4) that $q = F_p \equiv 1 \pmod{4}$. Hence, $H = \frac{1}{2}(q - 1)$ is even. We will consider two separate cases, but first we indicate some results which involve Pell numbers (and their "Lucas-Pell" counterparts):

(1)
$$a = 1 + \sqrt{2}, b = 1 - \sqrt{2};$$

(2)
$$P_n = \frac{a^n - b^n}{a - b}, \ Q_n = a^n + b^n, \ n = 0, \ 1, \ 2, \ \dots;$$

$$(3) \qquad P_{2n} = P_n Q_n;$$

 $Q_n^2 = Q_{2n} + 2(-1)^n$. (4)

Also, if P is an odd prime, the following congruences may be shown to be valid (see "Some Divisibility Properties of Generalized Fibonacci Sequences" by Paul S. Bruckman, The Fibonacci Quarterly 17.1 (1979), 42-49):

(5)
$$a^P \equiv a, b^P \equiv b \pmod{P}, \text{ iff } \left(\frac{2}{P}\right) = 1;$$

(6)
$$a^P \equiv b, b^P \equiv a \pmod{P}, \inf\left(\frac{2}{P}\right) = -1.$$

But (2 | P) = 1 iff $P \equiv \pm 1 \pmod{8}$; we may now complete the proof of the desired result.

Case I: $H \equiv 0 \pmod{4}$. Then $q = 2H + 1 \equiv 1 \pmod{8}$; using (5), we have $a^q \equiv a, b^q \equiv b \pmod{q}$,

so

$$a^{q-1} = a^{2H} \equiv b^{q-1} = b^{2H} \equiv 1 \pmod{q}$$

Hence,

(7)
$$P_{2H} \equiv 0, \ Q_{2H} \equiv 2 \pmod{q}$$
.

Also, using (3), (4), and (7), we have

 $P_{2H} = P_H Q_H \equiv 0 \pmod{q};$ (8)

(9)
$$Q_H^2 = Q_{2H} + 2 \equiv 4 \pmod{q}$$
.

Since $Q_H \neq 0$ (mod q) and $q \mid P_H Q_H$, it follows that $q \mid P_H$ in this case.

Case II:
$$H \equiv 2 \pmod{4}$$
. Then $q = 2H + 1 \equiv 5 \pmod{8}$. Hence, using (6),

$$a^q \equiv b, b^q \equiv a \pmod{q};$$

thus

(11)

$$a^{q+1} = a^{2H+2} \equiv b^{q+1} = b^{2H+2} \equiv -1 \pmod{q}$$

Therefore.

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 $P_{2H+2} \equiv 0$, $Q_{2H+2} \equiv -2 \pmod{q}$. (10)

Using (3), (4), and (10), we have

(11)
$$P_{2H+2} = P_{H+1}Q_{H+1} \equiv 0 \pmod{q};$$

(12)
$$Q_{H+1}^2 = Q_{2H+2} - 2 \equiv -4 \pmod{q}$$
.

Since $Q_{H+1} \neq 0 \pmod{q}$ and $q | P_{H+1}Q_{H+1}$, it follows that $q | P_{H+1}$. Q.E.D.

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Also solved by P. Tzermias and the proposers.

Two and Two Make ϕ

<u>H-429</u> Proposed by John Turner, Hamilton, New Zealand (Vol. 27, no. 1, February 1989)

Fibonacci enthusiasts know what happens when they add two adjacent numbers of a sequence and put the result next in line.

Have they considered what happens if they put the results in the middle? They will get the following increasing sequence of *T*-sets (multi-sets):

 $\left. \begin{array}{l} T_1 = \{1\} \\ T_2 = \{1, \ 2\} \end{array} \right\} \mbox{ given initial sets } \\ T_3 = \{1, \ 3, \ 2\}, \\ T_4 = \{1, \ 4, \ 3, \ 5, \ 2\}, \\ T_5 = \{1, \ 5, \ 4, \ 7, \ 3, \ 8, \ 5, \ 7, \ 2\}, \\ T_6 = \{1, \ 6, \ 5, \ 9, \ 4, \ 11, \ 7, \ 10, \ 3, \ 11, \ 8, \ 13, \ 5, \ 12, \ 7, \ 9, \ 2\}, \mbox{ etc.}$

Prove that for $3 \le i \le n$ the multiplicity of i in multi-set T_n is $\frac{1}{2}\phi(i)$, where ϕ is Euler's function.

Solution by the proposer

A binary tree can be grown, and rational numbers assigned to its nodes, as follows:

Assign (1/1) to the root node; then from each node in the tree grow a leftbranch and a right-branch and assign rational numbers to the new nodes as done below:



Assignment rule:

If $(p/q) = [a_0; a_1, a_2, \dots, a_n, 1]$ (simple continued fraction); (1/1) = [0; 1]; then assign

 $(p_1/q_1) = [a_0; a_1, a_2, \dots, a_n, 1, 1]$ (on left-branch node), and $(p_2/q_2) = [a_0; a_1, a_2, \dots, a_n + 1, 1]$ (on right-branch node).

It is easy to show [1] that all rational numbers are generated uniquely by this process (there is a one-to-one correspondence between the node values and the set of simple continued fractions whose last component is 1).

If the rational numbers (p/q) on the nodes in the left-hand subtree are considered, it will be seen that they will constitute the set of all rational numbers in the interval (0, 1) as the growth process continues ad infinitum. Hence, each q-value will occur $\phi(q)$ times, for q > 2.

The formation of the q-values in the tree, above the node (1/2), and in the left subtree from there corresponds to the formation of the integer values included in the T-sets at each stage.

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The right subtree about (1/2) generates an identical sequence of sets of q-values (in different order at each tree level).

The result of the problem follows immediately.

(Drawing the tree up to the fourth level will make all the above statements clear.)

Reference

1. A. G. Shaake & J. C. Turner. "A New Theory of Braiding (RR1/1)." Research Report No. 165 (1988), 1-42.

Also solved by P. Bruckman, S. Mohanty, and S. Shirali.

And More Identities

H-430 Proposed by Larry Taylor, Rego Park, NY (Vol. 27, no. 2, May 1989)

Find integers j, k (\neq 0, \pm 1, \pm 2), m_i and n_i such that:

- (A) $5F_{m_i}F_{n_i} = L_k + L_{j+i}$, for i = 1, 5, 9, 13, 17, 21;
- (B) $5F_{m_i}F_{n_i} = L_k L_{j+i}$, for i = 3, 7, 11, 15, 19, 23;
- (C) $F_{m_i} L_{n_i} = F_k + F_{j+i}$, for i = 1, 2, ..., 22, 23;
- (D) $L_{m_i}F_{n_i} = F_k F_{j+i}$, for i = 1, 3, ..., 21, 23;
- (E) $L_{m_i}L_{n_i} = L_k L_{j+i}$, for i = 1, 5, 9, 13, 17, 21;
- (F) $L_{m_{2}}L_{n_{2}} = L_{-k} + L_{j+i}$, for i = 2, 4, 6, 8;
- (G) $L_{m}L_{n} = L_{k} + L_{j+i}$, for i = 3, 7, 11, 15, 16, 18, 19, 20, 22, 23;
- (H) $L_{m_i}L_{n_i} = L_k + F_{j+i}$, for i = 10;
- (1) $L_{m_i}F_{n_i} = L_k + F_{j+i}$, for i = 12;
- (J) $5F_{m_i}F_{n_i} = L_k + F_{j+i}$, for i = 14.

Solution by Paul S. Bruckman, Edmonds, WA

Although there is some method to the process whereby j and k are discovered, there is also a lot of trial and error involved. It is easier to simply indicate, without further ado, the results of our search:

(1) j = -12, k = 7.

With these values, we find that the problem has solutions m_i and n_i , which are indicated below; no claim is made that other values of j and k cannot work equally well, though this seems likely.

(A)
$$L_7 + L_{-11} = 29 - 199 = -170 = 5(34)(-1) = 5F_9F_{-2};$$

 $L_7 + L_{-7} = 29 - 29 = 0 = 5F_7F_0;$
 $L_7 + L_{-3} = 29 - 4 = 25 = 5(5)(1) = 5F_5F_2;$
 $L_7 + L_1 = 29 + 1 = 30 = 5(2)(3) = 5F_3F_4;$
 $L_7 + L_5 = 29 + 11 = 40 = 5(1)(8) = 5F_1F_6;$
 $L_7 + L_9 = 29 + 76 = 105 = 5(1)(21) = 5F_{-1}F_8.$

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Note that we may take $m_i = \frac{1}{2}(19 - i)$, $n_i = 7 - n_i = \frac{1}{2}(i - 5)$, for all given i. (B) $L_7 - L_{-9} = 29 + 76 = 105 = 5(21)(1) = 5F_8F_{-1}$, etc., i.e., this yields the same results as part (A), in reverse order. With the same functions m_i and n_i as in part (A), we obtain the same identities. (C) $F_7 + F_{-11} = F_7 + F_{11} = 13 + 89 = 102 = 34 \cdot 3 = F_9L_2$ (*i* = 1, 23); $F_7 + F_{-10} = 13 - 55 = -42 = (-21)(2) = F_{-8}L_0$ (*i* = 2); $F_7 + F_{-9} = F_7 + F_9 = 13 + 34 = 47 = 1 \cdot 47 = F_1 L_8$ (*i* = 3, 21); $F_7 + F_{-8} = 13 - 21 = -8 = (2)(-4) = (-8)(1) = F_3L_{-3} = F_{-6}L_1$ (*i* = 4); $F_7 + F_{-7} = F_7 + F_7 = 13 + 13 = 26 = 13 \cdot 2 = F_7 L_0$ (*i* = 5, 19); $F_7 + F_{-6} = 13 - 8 = 5 = 5 \cdot 1 = F_5 L_1$ (*i* = 6); $F_7 + F_{-5} = F_7 + F_5 = 13 + 5 = 18 = 1 \cdot 18 = F_1L_6$ (*i* = 7, 17); $F_7 + F_{-4} = 13 - 3 = 10 = 5 \cdot 2 = F_5 L_0$ (*i* = 8); $F_7 + F_{-3} = F_7 + F_3 = 13 + 2 = 15 = 5 \cdot 3 = F_5 L_2$ (*i* = 9, 15); $F_7 + F_{-2} = 13 - 1 = 12 = 3 \cdot 4 = F_4 L_3$ (*i* = 10); $F_7 + F_{-1} = F_7 + F_1 = 13 + 1 = 14 = 2 \cdot 7 = F_3 L_4$ (*i* = 11, 13); $F_7 + F_0 = 13 = 13 \cdot 1 = F_7 L_1$ (*i* = 12); $F_7 + F_2 = 13 + 1 = 14 = F_3L_4$ (*i* = 14); $F_7 + F_4 = 13 + 3 = 16 = 8 \cdot 2 = F_6 L_0$ (*i* = 16); $F_7 + F_6 = 13 + 8 = 21 = 21 \cdot 1 = 3 \cdot 7 = F_8 L_1 = F_4 L_4$ (*i* = 18); $F_7 + F_8 = 13 + 21 = 34 = 34 \cdot 1 = F_9L_1$ (*i* = 20); $F_7 + F_{10} = 13 + 55 = 68 = 34 \cdot 2 = F_9 L_0$ (*i* = 22). (D) $F_7 - F_{-11} = F_7 - F_{11} = 13 - 89 = -76 = 76(-1) = L_9F_{-2}$ (*i* = 1, 23); $F_7 - F_{-9} = F_7 - F_9 = 13 - 34 = -21 = (-1)(21) = L_{-1}F_8$ (*i* = 3, 21); $F_7 - F_{-7} = F_7 - F_7 = 0 = L_7 F_0$ (*i* = 5, 19); $F_7 - F_{-5} = F_7 - F_5 = 13 - 5 = 8 = L_3F_3 = L_1F_6$ (*i* = 7, 17); $F_7 - F_{-3} = F_7 - F_3 = 13 - 2 = 11 = L_5 F_2$ (*i* = 9, 15); $F_7 - F_{-1} = F_7 - F_1 = 13 - 1 = 12 = 4 \cdot 3 = L_3 F_4$ (*i* = 11, 13). In this case, $m_i = \frac{1}{2}(19 - i)$, $n_i = \frac{1}{2}(i - 5)$, i = 1, 5, 9, 13, 17, 21; $m_i = \frac{1}{2}(i - 5), n_i = \frac{1}{2}(19 - i), i = 3, 7, 11, 15, 19, 23.$ (E) $L_7 - L_{-11} = 29 + 199 = 228 = 76 \cdot 3 = L_9 L_{-2};$ $L_7 - L_{-7} = 29 + 29 = 58 = 29 \cdot 2 = L_7 L_0;$ $L_7 - L_{-3} = 29 + 4 = 33 = 11 \cdot 3 = L_5 L_2;$ $L_7 - L_1 = 29 - 1 = 28 = 4 \cdot 7 = L_3 L_4;$ $L_7 - L_5 = 29 - 11 = 18 = 1 \cdot 18 = L_1 L_6$; $L_7 - L_9 = 29 - 76 = -47 = (-1)(47) = L_{-1}L_8.$ In this case, $m_i = \frac{1}{2}(19 - i)$, $n_i = \frac{1}{2}(i - 5)$.

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(F)
$$L_{-7} + L_{-10} = -29 + 123 = 94 = L_8 L_0$$
;
 $L_{-7} + L_{-8} = -29 + 47 = 18 = 18 \cdot 1 = L_6 L_1$;
 $L_{-7} + L_{-6} = -29 + 18 = -11 = 11(-1) = L_5 L_{-1}$;
 $L_{-7} + L_{-4} = -29 + 7 = -22 = (-11)(2) = L_{-5} L_0$.

(G)
$$L_7 + L_{-9} = 29 - 76 = -47 = (-1)(47) = L_{-1}L_8$$
;
 $L_7 + L_{-5} = 29 - 11 = 18 = 1 \cdot 18 = L_1L_6$;
 $L_7 + L_{-1} = 29 - 1 = 28 = 4 \cdot 7 = L_3L_4$;
 $L_7 + L_3 = 29 + 4 = 33 = 11 \cdot 3 = L_5L_2$;
 $L_7 + L_4 = 29 + 7 = 36 = 18 \cdot 2 = L_6L_0$;
 $L_7 + L_6 = 29 + 18 = 47 = 47 \cdot 1 = L_8L_1$;
 $L_7 + L_7 = 29 + 29 = 58 = 29 \cdot 2 = L_7L_0$;
 $L_7 + L_8 = 29 + 47 = 76 + 76 \cdot 1 = L_9L_1$;
 $L_7 + L_{10} = 29 + 123 = 152 = 76 \cdot 2 = L_9L_0$;
 $L_7 + L_{11} = 29 + 199 = 228 = 76 \cdot 3 = L_9L_2$.

(H)
$$L_7 + F_{-2} = 29 - 1 = 28 = 4 \cdot 7 = L_3 L_4$$
.

(I)
$$L_7 + F_0 = 29 + 0 = 29 \cdot 1 = L_7 F_1$$
.

(J)
$$L_7 + F_2 = 29 + 1 = 30 = 5 \cdot 2 \cdot 3 = 5F_3F_4$$
.

Also solved by L. Kuipers and the proposer.

Count to Five

<u>H-432</u> Proposed by Piero Filipponi, Rome, Italy (Vol. 27, no. 2, May 1989)

For k and n nonnegative integers and m a positive integer, let M(k, n, m) denote the arithmetic mean taken over the k^{th} powers of m consecutive Lucas numbers of which the smallest is L_n .

$$M(k, n, m) = \frac{1}{m} \sum_{j=n}^{n+m-1} L_j^k.$$

For $k = 2^h$ (h = 0, 1, 2, 3), find the smallest nontrivial value m_h ($m_h > 1$) of m for which M(k, n, m) is integral for every n.

Solution by the proposer

Let

$$L(k, n, m) = \sum_{j=n}^{n+m-1} L_j^k.$$

First, with the aid of Binet forms for F_s and L_s and use of the geometric series formula, we obtain the following general expression for L(2t, 0, s + 1) (t = 0, 1, ...):

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(1)
$$L(2t, 0, s+1) = \sum_{j=0}^{s} L_{j}^{2t} = {\binom{2t}{t}} X_{s,t} + \sum_{i=0}^{t-1} {\binom{2t}{i}} [(-1)^{is} L_{2(s+1)(t-i)} - (-1)^{i(s+1)} L_{2s(t-i)} + L_{2(t-i)} - 2(-1)^{i}] / [L_{2(t-i)} - 2(-1)^{i}],$$

where

(1')
$$X_{s,t} = \begin{cases} s+1 & \text{if } t \text{ is even,} \\ \\ [1+(-1)^s]/2 & \text{if } t \text{ is odd.} \end{cases}$$

Then, specializing (1) and (1') to t = 1, 2, and 4, after some simple but tedious manipulations involving the use of certain elementary Fibonacci identities (see V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*), we obtain

(2)
$$L(1, 0, s + 1) = L_{s+2} - 1;$$

(3) $L(2, 0, s + 1) = L_{2s+1} + 2 + (-1)^{s};$
(4) $L(4, 0, s + 1) = F_{4s+2} + 4(-1)^{s}F_{2s+1} + 6s + 11;$
(5) $L(8, 0, s + 1) = [F_{8s+4} + 84F_{4s+2} + 12(-1)^{s}(F_{6s+3} + 14F_{2s+1}) + 3(70s + 163)]/3,$

respectively. We point out that (2) has been obtained separately.

Case (i): k = 1 (h = 0)

From (2) we can write

(6) $L(1, n, m) = L(1, 0, n + m) - L(1, 0, n) = L_{n+m+1} - L_{n+1}$.

If m = 24, using Hoggatt's identities I_{24} and I_{32} , from (6) we can write

 $L(1, n, 24) = 5F_{12}F_{n+13}$

whence

(7)
$$M(1, n, 24) = L(1, n, 24)/24 = 30F_{n+13}$$

appears to be integral independently of n. Moreover, it can be readily verified that

M(1, 0, m) is not integral for m = 2, 4-23;

M(1, 1, 3) is not integral.

It follows that $m_0 = 24$.

Case (ii): k = 2 (h = 1)

From (3) we can write

(8) L(2, n, m) = L(2, 0, n + m) - L(2, 0, n)

$$= L_{2n+2m-1} - L_{2n-1} + (-1)^{n-m-1} - (-1)^{n-1}.$$

If m = 12, using Hoggatt's identities I_{24} and I_{32} , from (8) we can write

 $L(2, n, 12) = 5F_{12}F_{2n+11},$

whence

(9) $M(2, n, 12) = L(2, n, 12)/12 = 60F_{2n+11}$

appears to be integral independently of n. Moreover, it can be readily verified that

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M(2, 0, m) is not integral for m = 2-9, 11;M(2, 1, 10) is not integral.

It follows that $m_1 = 12$.

Case (iii): k = 4 (h = 2)

From (4) we can write

(10) L(4, n, m) = L(4, 0, n + m) - L(4, 0, n)

$$= F_{4n+4m-2} + 4(-1)^{m+n-1}F_{2n+2m-1} - 4(-1)^{n-1}F_{2n-1} + 6m.$$

If m = 5, using Hoggatt's identities I_{24} , I_{22} , and I_7 , (10) can be rewritten as $L(4, n, 5) = F_5[L_{4(n+2)}L_5 + 4(-1)^n L_{2(n+2)}] + 30$,

whence

(11)
$$M(4, n, 5) = L(4, n, 5)/5 = L_{4(n+2)}L_5 + 4(-1)^n L_{2(n+2)} + 6$$

appears to be integral independently of n. Moreover, it can be readily verified that

M(4, 0, m) is not integral for m = 2, 3, 4.

It follows that $m_2 = 5$.

Case (iv): k = 8 (h = 3)

Letting

(12) r = 2n + m - 1

and omitting the intermediate steps for brevity, from (5) we can write

(13) L(8, n, m) = L(8, 0, n + m) - L(8, 0, n)

 $= [L_{4p}F_{4m} + 84L_{2p}F_{2m} - 12(-1)^{n+m}(L_{3p}F_{3m} + 14L_{p}F_{m}) + 210m]/3$

 $= F_m [L_{4r} L_{2m} L_m + 84L_{2r} L_m - 12(-1)^{n+m} (L_{3r} F_{3m}/F_m + 14L_r)]/3 + 70m.$

Letting m = 5 in both (12) and (13), we have

$$\begin{split} L(8, n, 5) &= F_5 [1353 L_{8(n+2)} + 924 L_{4(n+2)} + 12 (-1)^n (122 L_{6(n+2)} \\ &+ 14 L_{2(n+2)})]/3 + 350 \\ &= 5 [451 L_{8(n+2)} + 308 L_{4(n+2)} + 4 (-1)^n (122 L_{6(n+2)} \\ &+ 14 L_{2(n+2)}] + 350, \end{split}$$

whence

(14) $M(8, n, 5) = L(8, n, 5)/5 = 451L_{8(n+2)} + 308L_{4(n+2)}$

$$+ 4(-1)^{n}(122L_{6(n+2)} + 14L_{2(n+2)}) + 70$$

appears to be integral independently of n. Moreover, it can be readily verified that

M(8, 0, m) is not integral for m = 2, 3, 4.

It follows that $m_3 = 5$.

Also solved by P. Bruckman.

<u>Editorial Note:</u> A number of readers have pointed out that H-440 and H-448 are essentially the same. Sorry about that.
