## ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

## PROBLEMS PROPOSED IN THIS ISSUE

H-449 Proposed by İoan Sadoveanu, Ellensburg, WA
Let $G(x)=x^{k}+a_{1} x^{k-1}+\ldots+\alpha_{k}$ be a polynomial with $c$ as a root of order p. If $G^{(p)}(x)$ denotes the $p^{\text {th }}$ derivative of $G(x)$, show that
$\left\{\frac{n^{p} c^{n-p}}{G^{(p)}(c)}\right\}$ is a solution to the recurrence
$u_{n}=c^{n-k}-a_{1} u_{n-1}-a_{2} u_{n-2}-\cdots-a_{k} u_{n-k}$.
H-450 Proposed by R. André-Jeannin, Sfax, Tunisia
Compare the numbers
$\theta=\sum_{n=1}^{\infty} \frac{1}{F_{n}}$
and
$\theta^{\prime}=2+\sum_{n=1}^{\infty} \frac{1}{F_{n}\left(2 F_{n-1}^{2}+(-1)^{n-1}\right)\left(2 F_{n}^{2}+(-1)^{n}\right)}$.
H-451 Proposed by T. V. Padmakumar, Trivandrum, South India
If $p$ is a prime and $x$ and $a$ are positive integers, show
$\binom{x+\alpha p}{p}-\binom{x}{p} \equiv a(\bmod p)$.

## SOLUTIONS

## Pell Mell

H-424 Proposed by Piero Filipponi \& Adina Di Porto, Rome, Italy (Vol. 26, no. 3, August 1988)

Let $F_{n}$ and $P_{n}$ denote the Fibonacci and Pell numbers, respectively.
Prove that, if $F_{p}$ is a prime $(p>3)$, then either $F_{p} \mid P_{H}$ or $F_{p} \mid P_{H+1}$, where $H=\left(F_{p}-1\right) / 2$.

Solution by Paul S. Bruckman, Edmonds, WA
Let $q=F_{p}>3$, a prime. Since $p \equiv \pm 1(\bmod 6)$, it is clear from a table of congruences (mod 4) that $q=F_{p} \equiv 1(\bmod 4)$. Hence, $H=\frac{1}{2}(q-1)$ is even. We will consider two separate cases, but first we indicate some results which involve Pell numbers (and their "Lucas-Pell" counterparts):

$$
\begin{equation*}
a=1+\sqrt{2}, b=1-\sqrt{2} \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
P_{n}=\frac{a^{n}-b^{n}}{a-b}, Q_{n}=a^{n}+b^{n}, n=0,1,2, \ldots \tag{2}
\end{equation*}
$$

$$
\begin{align*}
& P_{2 n}=P_{n} Q_{n}  \tag{3}\\
& Q_{n}^{2}=Q_{2 n}+2(-1)^{n}
\end{align*}
$$

Also, if $P$ is an odd prime, the following congruences may be shown to be valid (see "Some Divisibility Properties of Generalized Fibonacci Sequences" by Paul S. Bruckman, The Fibonacci Quarterly 17.1 (1979), 42-49):

$$
\begin{align*}
& a^{P} \equiv a, b^{P} \equiv b(\bmod P), \text { iff }\left(\frac{2}{P}\right)=1  \tag{5}\\
& a^{P} \equiv b, \quad b^{P} \equiv a(\bmod P), \text { iff }\left(\frac{2}{P}\right)=-1 \tag{6}
\end{align*}
$$

But $(2 \mid P)=1$ iff $P \equiv \pm 1(\bmod 8)$; we may now complete the proof of the desired result.

Case I: $H \equiv 0(\bmod 4)$. Then $q=2 H+1 \equiv 1(\bmod 8) ;$ using (5), we have $a^{q} \equiv a, b^{q} \equiv b(\bmod q)$,
so

$$
a^{q-1}=a^{2 H} \equiv b^{q-1}=b^{2 H} \equiv 1(\bmod q)
$$

Hence,

$$
\begin{equation*}
P_{2 H} \equiv 0, Q_{2 H} \equiv 2(\bmod q) \tag{7}
\end{equation*}
$$

Also, using (3), (4), and (7), we have

$$
\begin{align*}
& P_{2 H}=P_{H} Q_{H} \equiv 0(\bmod q)  \tag{8}\\
& Q_{H}^{2}=Q_{2 H}+2 \equiv 4(\bmod q) \tag{9}
\end{align*}
$$

Since $Q_{H} \not \equiv 0(\bmod q)$ and $q \mid P_{H} Q_{H}$, it follows that $q \mid P_{H}$ in this case.
Case II: $H \equiv 2(\bmod 4) . \quad$ Then $q=2 H+1 \equiv 5(\bmod 8)$. Hence, using (6), $a^{q} \equiv b, b^{q} \equiv a(\bmod q) ;$
thus

$$
a^{q+1}=a^{2 H+2} \equiv b^{q+1}=b^{2 H+2} \equiv-1(\bmod q)
$$

Therefore,
(10) $\quad P_{2 H+2} \equiv 0, Q_{2 H+2} \equiv-2(\bmod q)$.

Using (3), (4), and (10), we have
(12) $Q_{H+1}^{2}=Q_{2 H+2}-2 \equiv-4(\bmod q)$.

Since $Q_{H+1} \not \equiv 0(\bmod q)$ and $q \mid P_{H+1} Q_{H+1}$, it follows that $q \mid P_{H+1}$. Q.E.D.

Also solved by P. Tzermias and the proposers.
Two and Two Make $\phi$
H-429 Proposed by John Turner, Hamilton, New Zealand (Vol. 27, no. 1, February 1989)

Fibonacci enthusiasts know what happens when they add two adjacent numbers of a sequence and put the result next in line.

Have they considered what happens if they put the results in the middle?
They will get the following increasing sequence of $T$-sets (multi-sets):

$$
\left.\begin{array}{l}
T_{1}=\{1\} \\
T_{2}=\{1,2\}
\end{array}\right\} \text { given initial sets }, ~ \begin{aligned}
& T_{3}=\{1,3,2\}, \\
& T_{4}=\{1,4,3,5,2\}, \\
& T_{5}=\{1,5,4,7,3,8,5,7,2\}, \\
& T_{6}=\{1,6,5,9,4,11,7,10,3,11,8,13,5,12,7,9,2\}, \\
& \text { etc. }
\end{aligned}
$$

Prove that for $3 \leq i \leq n$ the multiplicity of $i$ in multi-set $T_{n}$ is $\frac{1}{2} \phi(i)$, where $\phi$ is Euler's function.

Solution by the proposer
A binary tree can be grown, and rational numbers assigned to its nodes, as follows:

Assign (1/1) to the root node; then from each node in the tree grow a leftbranch and a right-branch and assign rational numbers to the new nodes as done below:


Assignment rule:
If $(p / q)=\left[\alpha_{0} ; \alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}, 1\right]$ (simple continued fraction);
$(1 / 1)=[0 ; 1]$; then assign

$$
\left(p_{1} / q_{1}\right)=\left[a_{0} ; a_{1}, a_{2}, \ldots, a_{n}, 1,1\right] \text { (on 1eft-branch node) }
$$

and $\left(p_{2} / q_{2}\right)=\left[a_{0} ; a_{1}, a_{2}, \ldots, a_{n}+1,1\right]$ (on right-branch node).
It is easy to show [1] that all rational numbers are generated uniquely by this process (there is a one-to-one correspondence between the node values and the set of simple continued fractions whose last component is 1).

If the rational numbers $(p / q)$ on the nodes in the left-hand subtree are considered, it will be seen that they will constitute the set of all rational numbers in the interval $(0,1)$ as the growth process continues ad infinitum. Hence, each $q$-value will occur $\phi(q)$ times, for $q>2$.

The formation of the $q$-values in the tree, above the node (1/2), and in the left subtree from there corresponds to the formation of the integer values included in the $T$-sets at each stage.

The right subtree about (1/2) generates an identical sequence of sets of $q$-values (in different order at each tree level).

The result of the problem follows immediately.
(Drawing the tree up to the fourth level will make all the above statements clear.)

## Reference

1. A. G. Shaake \& J. C. Turner. "A New Theory of Braiding (RR1/1)." Research Report No. 165 (1988), 1-42.

Also solved by P. Bruckman, S. Mohanty, and S. Shirali.

## And More Identities

H-430 Proposed by Larry Taylor, Rego Park, NY
(Vol. 27, no. 2, May 1989)
Find integers $j, k(\neq 0, \pm 1, \pm 2), m_{i}$ and $n_{i}$ such that:
(A) $5 F_{m_{i}} F_{n_{i}}=L_{k}+L_{j+i}$, for $i=1,5,9,13,17,21$;
(B) $5 F_{m_{i}} F_{n_{i}}=L_{k}-L_{j+i}$, for $i=3,7,11,15,19,23$;
(C) $F_{m_{i}} L_{n_{i}}=F_{k}+F_{j+i}$, for $i=1,2, \ldots, 22,23$;
(D) $L_{m_{i}} F_{n_{i}}=F_{k}-F_{j+i}$, for $i=1,3, \ldots, 21,23$;
(E) $L_{m_{i}} L_{n_{i}}=L_{k}-L_{j+i}$, for $i=1,5,9,13,17,21$;
(F) $L_{m_{i}} L_{n_{i}}=L_{-k}+L_{j+i}$, for $i=2,4,6,8$;
(G) $L_{m_{i}} L_{n_{i}}=L_{k}+L_{j+i}$, for $i=3,7,11,15,16,18,19,20,22,23 ;$
(H) $L_{m_{i}} L_{n_{i}}=L_{k}+F_{j+i}$, for $i=10$;
(I) $L_{m_{i}} F_{n_{i}}=L_{k}+F_{j+i}$, for $i=12$;
(J) $5 F_{m_{i}} F_{n_{i}}=L_{k}+F_{j+i}$, for $i=14$.

Solution by Paul S. Bruckman, Edmonds, WA
Although there is some method to the process whereby $j$ and $k$ are discovered, there is also a lot of trial and error involved. It is easier to simply indicate, without further ado, the results of our search:
(1) $j=-12, k=7$.

With these values, we find that the problem has solutions $m_{i}$ and $n_{i}$, which are indicated below; no claim is made that other values of $j$ and $k$ cannot work equally well, though this seems likely.
(A) $L_{7}+L_{-11}=29-199=-170=5(34)(-1)=5 F_{9} F_{-2}$;
$L_{7}+L_{-7}=29-29=0=5 F_{7} F_{0} ;$
$L_{7}+L_{-3}=29-4=25=5(5)(1)=5 F_{5} F_{2}$;
$L_{7}+L_{1}=29+1=30=5(2)(3)=5 F_{3} F_{4}$;
$L_{7}+L_{5}=29+11=40=5(1)(8)=5 F_{1} F_{6}$;
$L_{7}+L_{9}=29+76=105=5(1)(21)=5 F_{-1} F_{8}$.

Note that we may take $m_{i}=\frac{1}{2}(19-i), n_{i}=7-n_{i}=\frac{1}{2}(i-5)$, for all given $i$. (B) $L_{7}-L_{-9}=29+76=105=5(21)(1)=5 F_{8} F_{-1}$, etc.,
i.e., this yields the same results as part (A), in reverse order. With the same functions $m_{i}$ and $n_{i}$ as in part (A), we obtain the same identities.
(C) $F_{7}+F_{-11}=F_{7}+F_{11}=13+89=102=34 \cdot 3=F_{9} L_{2}(i=1,23)$;
$F_{7}+F_{-10}=13-55=-42=(-21)(2)=F_{-8} L_{0} \quad(i=2) ;$
$F_{7}+F_{-9}=F_{7}+F_{9}=13+34=47=1 \cdot 47=F_{1} L_{8}(i=3,21) ;$
$F_{7}+F_{-8}=13-21=-8=(2)(-4)=(-8)(1)=F_{3} L_{-3}=F_{-6} L_{1} \quad(i=4) ;$
$F_{7}+F_{-7}=F_{7}+F_{7}=13+13=26=13 \cdot 2=F_{7} L_{0} \quad(i=5,19) ;$
$F_{7}+F_{-6}=13-8=5=5 \cdot 1=F_{5} L_{1} \quad(i=6)$;
$F_{7}+F_{-5}=F_{7}+F_{5}=13+5=18=1 \cdot 18=F_{1} L_{6}(i=7,17) ;$
$F_{7}+F_{-4}=13-3=10=5 \cdot 2=F_{5} L_{0} \quad(i=8) ;$
$F_{7}+F_{-3}=F_{7}+F_{3}=13+2=15=5 \cdot 3=F_{5} L_{2}(i=9,15) ;$
$F_{7}+F_{-2}=13-1=12=3 \cdot 4=F_{4} L_{3} \quad(i=10) ;$
$F_{7}+F_{-1}=F_{7}+F_{1}=13+1=14=2 \cdot 7=F_{3} L_{4}(i=11,13) ;$
$F_{7}+F_{0}=13=13 \cdot 1=F_{7} L_{1} \quad(i=12) ;$
$F_{7}+F_{2}=13+1=14=F_{3} L_{4} \quad(i=14) ;$
$F_{7}+F_{4}=13+3=16=8 \cdot 2=F_{6} L_{0} \quad(i=16)$;
$F_{7}+F_{6}=13+8=21=21 \cdot 1=3 \cdot 7=F_{8} L_{1}=F_{4} L_{4} \quad(i=18) ;$
$F_{7}+F_{8}=13+21=34=34 \cdot 1=F_{9} L_{1} \quad(i=20) ;$
$F_{7}+F_{10}=13+55=68=34 \cdot 2=F_{9} L_{0} \quad(i=22)$ 。
(D) $F_{7}-F_{-11}=F_{7}-F_{11}=13-89=-76=76(-1)=L_{9} F_{-2}(i=1,23)$;
$F_{7}-F_{-9}=F_{7}-F_{9}=13-34=-21=(-1)(21)=L_{-1} F_{8}(i=3,21) ;$
$F_{7}-F_{-7}=F_{7}-F_{7}=0=L_{7} F_{0}(i=5,19) ;$
$F_{7}-F_{-5}=F_{7}-F_{5}=13-5=8=L_{3} F_{3}=L_{1} F_{6}(i=7,17)$;
$F_{7}-F_{-3}=F_{7}-F_{3}=13-2=11=L_{5} F_{2}(i=9,15) ;$
$F_{7}-F_{-1}=F_{7}-F_{1}=13-1=12=4 \cdot 3=L_{3} F_{4}(i=11,13)$.
In this case, $m_{i}=\frac{1}{2}(19-i), n_{i}=\frac{1}{2}(i-5), i=1,5,9,13,17,21$;
$m_{i}=\frac{1}{2}(i-5), n_{i}=\frac{1}{2}(19-i), i=3,7,11,15,19,23$.
(E) $L_{7}-L_{-11}=29+199=228=76 \cdot 3=L_{9} L_{-2}$;
$L_{7}-L_{-7}=29+29=58=29 \cdot 2=L_{7} L_{0} ;$
$L_{7}-L_{-3}=29+4=33=11 \cdot 3=L_{5} L_{2}$;
$L_{7}-L_{1}=29-1=28=4 \cdot 7=L_{3} L_{4}$;
$L_{7}-L_{5}=29-11=18=1 \cdot 18=L_{1} L_{6}$;
$L_{7}-L_{9}=29-76=-47=(-1)(47)=L_{-1} L_{8}$.
In this case, $m_{i}=\frac{1}{2}(19-i), n_{i}=\frac{1}{2}(i-5)$.
(F) $L_{-7}+L_{-10}=-29+123=94=L_{8} L_{0}$;
$L_{-7}+L_{-8}=-29+47=18=18 \cdot 1=L_{6} L_{1} ;$
$L_{-7}+L_{-6}=-29+18=-11=11(-1)=L_{5} L_{-1} ;$
$L_{-7}+L_{-4}=-29+7=-22=(-11)(2)=L_{-5} L_{0}$.
(G) $L_{7}+L_{-9}=29-76=-47=(-1)(47)=L_{-1} L_{8}$;
$L_{7}+L_{-5}=29-11=18=1 \cdot 18=L_{1} L_{6} ;$
$L_{7}+L_{-1}=29-1=28=4 \cdot 7=L_{3} L_{4}$;
$L_{7}+L_{3}=29+4=33=11 \cdot 3=L_{5} L_{2} ;$
$L_{7}+L_{4}=29+7=36=18 \cdot 2=L_{6} L_{0} ;$
$L_{7}+L_{6}=29+18=47=47 \cdot 1=L_{8} L_{1} ;$
$L_{7}+L_{7}=29+29=58=29 \cdot 2=L_{7} L_{0}$;
$L_{7}+L_{8}=29+47=76+76 \cdot 1=L_{9} L_{1} ;$
$L_{7}+L_{10}=29+123=152=76 \cdot 2=L_{9} L_{0}$;
$L_{7}+L_{11}=29+199=228=76 \cdot 3=L_{9} L_{2}$.
(H) $L_{7}+F_{-2}=29-1=28=4 \cdot 7=L_{3} L_{4}$.
(I) $L_{7}+F_{0}=29+0=29 \cdot 1=L_{7} F_{1}$.
(J) $L_{7}+F_{2}=29+1=30=5 \cdot 2 \cdot 3=5 F_{3} F_{4}$.

Also solved by L. Kuipers and the proposer.

## Count to Five

H-432 Proposed by Piero Filipponi, Rome, Italy (Vol. 27, no. 2, May 1989)

For $k$ and $n$ nonnegative integers and $m$ a positive integer, let $M(k, n, m)$ denote the arithmetic mean taken over the $k^{\text {th }}$ powers of $m$ consecutive Lucas numbers of which the smallest is $L_{n}$.

$$
M(k, n, m)=\frac{1}{m} \sum_{j=n}^{n+m-1} L_{j}^{k} .
$$

For $k=2^{h}(h=0,1,2,3)$, find the smallest nontrivial value $m_{h}\left(m_{h}>1\right)$ of $m$ for which $M(k, n, m)$ is integral for every $n$.

Solution by the proposer
Let

$$
L(k, n, m)=\sum_{j=n}^{n+m-1} L_{j}^{k} .
$$

First, with the aid of Binet forms for $F_{s}$ and $L_{s}$ and use of the geometric series formula, we obtain the following general expression for $L(2 t, 0, s+1)$ ( $t=0,1, \ldots$ ):

$$
\begin{align*}
& L(2 t, 0, s+1)=\sum_{j=0}^{s} L_{j}^{2 t}=\binom{2 t}{t} X_{s, t}+\sum_{i=0}^{t-1}\binom{2 t}{i}\left[(-1)^{i s} L_{2(s+1)(t-i)}\right.  \tag{1}\\
& \left.-(-1)^{i(s+1)} L_{2 s(t-i)}+L_{2(t-i)}-2(-1)^{i}\right] /\left[L_{2(t-i)}-2(-1)^{i}\right]
\end{align*}
$$

where
$\left(1^{\prime}\right) \quad X_{s, t}= \begin{cases}s+1 & \text { if } t \text { is even, } \\ {\left[1+(-1)^{s}\right] / 2} & \text { if } t \text { is odd. }\end{cases}$
Then, specializing (1) and ( $1^{\prime}$ ) to $t=1,2$, and 4 , after some simple but tedious manipulations involving the use of certain elementary Fibonacci identities (see V. E. Hoggatt, Jr., Fibonacci and Lucas Numbers), we obtain
(2) $\quad L(1,0, s+1)=L_{s+2}-1$;
(3) $\quad L(2,0, s+1)=L_{2 s+1}+2+(-1)^{s}$;
(4) $\quad L(4,0, s+1)=F_{4 s+2}+4(-1)^{s} F_{2 s+1}+6 s+11$;
(5) $\quad L(8,0, s+1)=\left[F_{8 s+4}+84 F_{4 s+2}+12(-1)^{s}\left(F_{6 s+3}+14 F_{2 s+1}\right)\right.$

$$
+3(70 s+163)] / 3
$$

respectively. We point out that (2) has been obtained separately.
Case (i): $k=1(h=0)$
From (2) we can write
(6) $\quad L(1, n, m)=L(1,0, n+m)-L(1,0, n)=L_{n+m+1}-L_{n+1}$.

If $m=24$, using Hoggatt's identities $I_{24}$ and $I_{32}$, from (6) we can write $L(1, n, 24)=5 F_{12} F_{n+13}$
whence
$M(1, n, 24)=L(1, n, 24) / 24=30 F_{n+13}$
appears to be integral independently of $n$. Moreover, it can be readily verified that
$M(1,0, m)$ is not integral for $m=2,4-23$;
$M(1,1,3)$ is not integral.
It follows that $m_{0}=24$.
Case (ii): $k=2(h=1)$
From (3) we can write

$$
\begin{equation*}
L(2, n, m)=L(2,0, n+m)-L(2,0, n) \tag{8}
\end{equation*}
$$

$$
=L_{2 n+2 m-1}-L_{2 n-1}+(-1)^{n-m-1}-(-1)^{n-1}
$$

If $m=12$, using Hoggatt's identities $I_{24}$ and $I_{32}$, from (8) we can write

$$
L(2, n, 12)=5 F_{12} F_{2 n+11},
$$

whence
(9) $\quad M(2, n, 12)=L(2, n, 12) / 12=60 F_{2 n+11}$
appears to be integral independently of $n$. Moreover, it can be readily verified that

```
    M(2, 0, m) is not integral for m=2-9, 11;
    M(2, 1, 10) is not integral.
    It follows that m}\mp@subsup{m}{1}{}=12\mathrm{ .
    Case (iii): k=4 (h=2)
```

From (4) we can write

$$
\begin{equation*}
L(4, n, m)=L(4,0, n+m)-L(4,0, n) \tag{10}
\end{equation*}
$$

$$
=F_{4 n+4 m-2}+4(-1)^{m+n-1} F_{2 n+2 m-1}-4(-1)^{n-1} F_{2 n-1}+6 m
$$

If $m=5$, using Hoggatt's identities $I_{24}, I_{22}$, and $I_{7},(10)$ can be rewritten as $L(4, n, 5)=F_{5}\left[L_{4(n+2)} L_{5}+4(-1)^{n} L_{2(n+2)}\right]+30$,
whence
$M(4, n, 5)=L(4, n, 5) / 5=L_{4(n+2)} L_{5}+4(-1)^{n} L_{2(n+2)}+6$
appears to be integral independently of $n$. Moreover, it can be readily verified that
$M(4,0, m)$ is not integral for $m=2,3,4$.
It follows that $m_{2}=5$.
Case (iv): $k=8(h=3)$
Letting

$$
\begin{equation*}
r=2 n+m-1 \tag{12}
\end{equation*}
$$

and omitting the intermediate steps for brevity, from (5) we can write

$$
\begin{equation*}
L(8, n, m)=L(8,0, n+m)-L(8,0, n) \tag{13}
\end{equation*}
$$

$=\left[L_{4 r} F_{4 m}+84 L_{2 r} F_{2 m}-12(-1)^{n+m}\left(L_{3 r} F_{3 m}+14 L_{r} F_{m}\right)+210 m\right] / 3$
$=F_{m}\left[L_{4 r} L_{2 m} L_{m}+84 L_{2 r} L_{m}-12(-1)^{n+m}\left(L_{3 r} F_{3 m} / F_{m}+14 L_{r}\right)\right] / 3+70 m$.
Letting $m=5$ in both (12) and (13), we have

$$
\begin{aligned}
& L(8, n, 5)= F_{5}\left[1353 L_{8(n+2)}+924 L_{4(n+2)}+12(-1)^{n}\left(122 L_{6(n+2)}\right.\right. \\
&\left.\left.+14 L_{2(n+2)}\right)\right] / 3+350 \\
&=5\left[451 L_{8(n+2)}+308 L_{4(n+2)}+4(-1)^{n}\left(122 L_{6(n+2)}\right.\right. \\
&\left.+14 L_{2(n+2)}\right]+350,
\end{aligned}
$$

whence

$$
\begin{align*}
M(8, n, 5)=L(8, n, 5) / 5= & 451 L_{8(n+2)}+308 L_{4(n+2)}  \tag{14}\\
& +4(-1)^{n}\left(122 L_{6(n+2)}+14 L_{2(n+2)}\right)+70
\end{align*}
$$

appears to be integral independently of $n$. Moreover, it can be readily verified that
$M(8,0, m)$ is not integral for $m=2,3,4$.
It follows that $m_{3}=5$.
Also solved by $P$. Bruckman.
Editorial Note: A number of readers have pointed out that $\mathrm{H}-440$ and $\mathrm{H}-448$ are essentially the same. Sorry about that.

