GENERALIZED COMPLEX FIBONACCI AND LUCAS FUNCTIONS

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(Submitted December 1988)

1. Introduction

Eric Halsey [3] has invented a method for defining the Fibonacci numbers $F(x)$, where x is a real number. Unfortunately, the Fibonacci identity

(1) $$F(x) = F(x - 1) + F(x - 2)$$

is destroyed. We shall return later to his method.

Francis Parker [6] defines the Fibonacci function by

$$F(x) = \frac{\alpha^x - \cos \pi x \alpha^{-x}}{\sqrt{5}},$$

where α is the golden ratio. In the same way, we can define a Lucas function

$$L(x) = \alpha^x + \cos \pi x \alpha^{-x}.$$

$F(x)$ and $L(x)$ coincide with the usual Fibonacci and Lucas numbers when x is an integer, and the relation (1) is verified. But the classical Fibonacci relations do not generalize. For instance, we do not have

$$F(2x) = F(x)L(x).$$

Horadam and Shannon [4] define Fibonacci and Lucas curves. They can be written, with complex notation

(2) $$F(x) = \frac{\alpha^x - e^{i\pi x}\alpha^{-x}}{\sqrt{5}},$$

(3) $$L(x) = \alpha^x + e^{i\pi x}\alpha^{-x}.$$

Again, we have $F(n) = F_n$, $L(n) = L_n$, for all integers n.

We shall prove in the sequel that the well-known identities for F_n and L_n are again true for all real numbers x, if $F(x)$ and $L(x)$ are defined by (2) and (3). For example, we have immediately

$$F(2x) = F(x)L(x).$$

We shall also relate these $F(x)$ and $L(x)$ to other Fibonacci properties as well as to Halsey's extension of the Fibonacci numbers.

2. Preliminary Lemma

Let us consider the set E of functions $w : \mathbb{R} \to \mathbb{C}$ such that

(4) $$\forall x \in \mathbb{R},\ w(x) = w(x - 1) + w(x - 2).$$

E is a complex vector space, and the following lemma is immediate.

Lemma 1: Let α be the positive root of $r^2 = r + 1$. Then the functions f and g, defined by

$$f(x) = \alpha^x, \quad g(x) = e^{i\pi x}\alpha^{-x}$$

are members of E.

Let us define now a subspace V of E by

$$V = \{w : \mathbb{R} \to \mathbb{C},\ w = \lambda f + \mu g,\ \lambda,\ \mu \in \mathbb{C}\}.$$

The functions F and L, defined by (2) and (3), are members of V.

Lemma 2: For all complex numbers a and b, there is a unique function w in V such that

$$w(0) = a, \quad w(1) = b.$$

Proof: We have

$$w(0) = \lambda + \mu = a, \quad w(1) = \lambda\alpha - \mu\alpha^{-1} = b.$$

By Cramer's rule, λ and μ exist and are unique.

Lemma 3: Let w be a member of V, and h a real number. Then the functions w_h and w_h', defined by

$$w_h(x) = w(x - h), \quad w_h'(x) = e^{i\pi x}w(h - x),$$

are members of V.

Proof: The proof is simple and therefore is omitted here.

Lemma 4: Let u and v be two elements of V and $\delta : \mathbb{R}^2 \to \mathbb{C}$, the function defined by

$$\delta(x, y) = \begin{vmatrix} u(x), & u(x+1) \\ v(y), & v(y+1) \end{vmatrix} = u(x)v(y+1) - u(x+1)v(y).$$

Then we have

(5) $$\delta(x, y) = e^{i\pi y}\delta(x - y, 0).$$

Proof: First, we have

(6) $$\delta(x, y) = \begin{vmatrix} u(x), & u(x) + u(x-1) \\ v(y), & v(y) + v(y-1) \end{vmatrix} = \begin{vmatrix} u(x), & u(x-1) \\ v(y), & v(y-1) \end{vmatrix}$$

$$= -\delta(x - 1, y - 1).$$

Now, let us define

$$\eta(x, y) = e^{i\pi y}\delta(x - y, 0) = e^{i\pi y}(u(x - y)v(1) - u(x - y + 1)v(0)).$$

Let x be a fixed real number. By Lemma 3, the functions

$$y \to \delta(x, y), \quad y \to \eta(x, y)$$

are members of V. We have

$$\delta(x, 0) = \eta(x, 0),$$

and, by (6),

$$\delta(x, 1) = -\delta(x - 1, 0) = \eta(x, 1).$$

By Lemma 2 we have, for all real numbers y,

$$\delta(x, y) = \eta(x, y).$$

This concludes the proof.

Lemma 5: Let F and L be the Fibonacci and Lucas functions defined by (2) and (3). Then, for all real numbers, we have:

(7) $$L(x) = F(x + 1) + F(x - 1);$$

(8) $5F(x) = 2L(x + 1) - L(x);$

(9) $L(x) = 2F(x + 1) - F(x).$

The proofs readily follow from the lemmas and the definitions of the functions.

3. The Main Result

Theorem 1: Let u and v be two functions of V. Then, for all values of x, y, and z, we have

(10) $u(x)v(y+z) - u(x+z)v(y) = e^{i\pi y}F(z)[u(x-y)v(1) - u(x-y+1)v(0)],$

where F is defined by (2).

Proof: For x and y fixed, consider the function Δ:

$$\Delta(z) = u(x)v(y + z) - u(x + z)v(y).$$

By Lemma 3, Δ is a member of V, and we have, with the notation of Lemma 4,

$$\Delta(0) = 0, \quad \Delta(1) = \delta(x, y).$$

Thus, we have, since the two members take the same values at $z = 0$, $z = 1$:

$$\Delta(z) = \delta(x, y)F(z).$$

The proof follows by Lemma 4.

4. Special Cases

Let us examine some particular cases of (10):

Case 1. $u = v = F$

Since $F(0) = 0$, $F(1) = 1$, we have

(11) $F(x)F(y + z) - F(x + z)F(y) = e^{i\pi y}F(z)F(x - y).$

Case 2. $u = v = L$

Since $L(0) = 2$, $L(1) = 1$, we have, by (8),

(12) $L(x)L(y + z) - L(x + z)L(y) = -5e^{i\pi y}F(z)F(x - y).$

Case 3. $u = F$, $v = L$

We have, by (9),

(13) $F(x)L(y + z) - F(x + z)L(y) = -e^{i\pi y}F(z)L(x - y).$

Case 4. $u = L$, $v = F$

(14) $L(x)F(y + z) - L(x + z)F(y) = e^{i\pi y}F(z)L(x - y).$

Case 5. Let $y = 0$ in (12) and (13) to get

(15) $2L(x + z) = L(x)L(z) + 5F(x)F(z),$

(16) $2F(x + z) = F(x)L(z) + F(z)L(x).$

Case 6. Let $y = 1$ in (11)-(14) to get

(17) $F(x + z) = F(x)F(z + 1) + F(z)F(x - 1),$

(18) $L(x + z) = L(x)L(z + 1) - 5F(z)F(x - 1),$

(19) $F(x + z) = F(x)L(z + 1) - F(z)L(x - 1),$

(20) $L(x + z) = L(x)F(z + 1) + F(z)L(x - 1).$

Case 7. Let $y = x - z$ in (11)-(14) to get

(21) $$(F(x))^2 - F(x + z)F(x - z) = e^{i\pi(x - z)}(F(z))^2,$$

(22) $$(L(x))^2 - L(x + z)L(x - z) = -5e^{i\pi(x - z)}(F(z))^2,$$

(23) $$F(x)L(x) - F(x + z)L(x - z) = -e^{i\pi(x - z)}F(z)L(z),$$

(24) $$F(x)L(x) - F(x - z)L(x + z) = e^{i\pi(x - z)}F(z)L(z).$$

Remark: (21) and (22) are Catalan's relations for $F(x)$, $L(x)$.

5. Application: A Reciprocal Series of Fibonacci Numbers

Theorem 2: Let x be a strictly positive real number and F the Fibonacci function. Then we have

$$\sum_{k=1}^{\infty} \frac{e^{i\pi 2^{k-1}x}}{F(x \cdot 2^k)} = \frac{e^{i\pi x}}{F(x)\alpha^x}.$$

Proof: We recall the relation attributed to De Morgan by Bromwich and to Catalan by Lucas,

(25) $$\sum_{k=1}^{n} \frac{z^{2^{k-1}}}{1 - z^{2^k}} = \frac{1}{1 - z}\,\frac{z - z^{2^n}}{1 - z^{2^n}},$$

where z is a complex number ($|z| \neq 1$). Now put $z = e^{i\pi x}\alpha^{-2x}$ in (25) to obtain:

(26) $$\sum_{k=1}^{n} \frac{e^{i\pi 2^{k-1}x}\alpha^{-2^k x}}{1 - e^{i\pi 2^k x}\alpha^{-2^{k+1}x}} = \sum_{k=1}^{n} \frac{e^{i\pi 2^{k-1}x}}{\alpha^{2^k x} - e^{i\pi 2^k x}\alpha^{-2^k x}} = \frac{1}{\sqrt{5}}\sum_{k=1}^{n} \frac{e^{i\pi 2^{k-1}x}}{F(2^k x)}$$

On the other hand, the right member of (25) becomes

(27) $$\frac{1}{1 - e^{i\pi x}\alpha^{-2x}} \cdot \frac{e^{i\pi x}\alpha^{-2x} - e^{i\pi 2^n x}\alpha^{-2^{n+1}x}}{1 - e^{i\pi 2^n x}\alpha^{-2^{n+1}x}} = \frac{1}{\sqrt{5}F(x)} \cdot \frac{e^{i\pi x}F((2^n - 1)x)}{F(x \cdot 2^n)}.$$

(26) and (27) give us

(28) $$\sum_{k=1}^{n} \frac{e^{i\pi 2^{k-1}x}}{F(2^k x)} = \frac{e^{i\pi x}F((2^n - 1)x)}{F(2^n \cdot x)F(x)},$$

and so

(29) $$\sum_{k=1}^{\infty} \frac{e^{i\pi 2^{k-1}x}}{F(2^k x)} = \frac{e^{i\pi x}}{F(x)\alpha^x}.$$

Remark: Put $x = m$ in (29), where m is a natural integer. After some calculations in the case m odd, we obtain the well-known formula:

(30) $$\sum_{k=1}^{\infty} \frac{1}{F(2^k m)} = \frac{\sqrt{5}}{\alpha^{2m} - 1}.$$

Formula (30) was found by Lucas (see [5], p. 225) and was rediscovered by Brady [1]. See also Gould [2] for complete references.

6. Halsey's Fibonacci Function

First, we recall a well-known formula,

$$F_n = \sum_{k=0}^{m(n)} \binom{n - k - 1}{k}, \quad n \geq 1,$$

where $m(n)$ is an integer such that $(n/2) - 1 \leq m(n) < (n/2)$.

We have used the binomial coefficients $\binom{n}{k}$ only when n is a positive integer but it is very convenient to extend their definitions. Then

$$\binom{x}{0} = 1, \quad \binom{x}{k} = \frac{x(x-1)\ \dots\ (x-k+1)}{k!}, \ k \geq 1,$$

defines the binomial coefficients for all values of x.

From this, we can introduce the function G,

$$G(x) = \sum_{k=0}^{m(x)} \binom{x-k-1}{k}, \ x > 0, \tag{31}$$

where $m(x)$ is the integer defined by $(x/2) - 1 \leq m(x) < (x/2)$. Then, clearly, we have

$$G(n) = F_n, \ n \geq 1.$$

Theorem 3: G coincides with Halsey's extension of Fibonacci numbers, namely,

$$G(x) = \sum_{k=0}^{m(x)} [(x-k)B(x-2k, k+1)]^{-1}, \ x > 0,$$

where $B(x, y)$ is the beta-function:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x > 0, \ y > 0.$$

Proof: It is sufficient to show that

$$\frac{1}{(x-k)B(x-2k, k+1)} = \binom{x-k-1}{k}. \tag{32}$$

In fact, the left member of (32) is

$$\frac{\Gamma(x-k+1)}{(x-k)\Gamma(x-2k)\Gamma(k+1)} = \frac{(x-k)(x-k-1)\ \dots\ (x-2k)\Gamma(x-2k)}{(x-k)\Gamma(x-2k)k!}$$
$$= \frac{(x-k-1)\ \dots\ (x-2k)}{k!} = \binom{x-k-1}{k},$$

in which we have used the well-known properties of the gamma-function:

$$\Gamma(x) = (x-1)\Gamma(x-1), \quad \Gamma(k) = (k-1)!$$

This concludes the proof.

Let p be a positive integer, and let G_p be the polynomial defined by

$$G_p(x) = \sum_{k=0}^{p} \binom{x-k-1}{k}.$$

We see, from (31), that

$$G(x) = G_p(x), \quad 2p < x \leq 2p + 2; \tag{33}$$

thus,

$$G_p(2p+1) = G(2p+1) = F_{2p+1},$$
$$G_p(2p+2) = G(2p+2) = F_{2p+2}.$$

In fact, we have a deeper result, which we state as the following theorem.

Theorem 4: $G_p(n) = F_n$ for $n = p + 1, p + 2, \dots, 2p + 2$.

Proof: We shall prove this by mathematical induction. If $p = 0$, we have

$$G_0(1) = G_0(2) = 1.$$

Now we suppose that $G_{p-1}(n) = F_n$ $(n = p, \dots, 2p)$. Then we have

$$G_p(x) = G_{p-1}(x) + \binom{x-p-1}{p} = G_{p-1}(x) + \frac{(x-p-1)\ \dots\ (x-2p)}{p!},$$

and thus,

$$G_p(n) = G_{p-1}(n) = F_n, \text{ for } n = p+1, \dots, 2p;$$

but we have seen above that

$$G_p(2p+1) = F_{2p+1}, \quad G_p(2p+2) = F_{2p+2}.$$

This concludes the proof.

Corollary: G is continuous for all values of $x > 0$.

Proof: By (33), it is sufficient to show the continuity from the right at $x = 2p$. But

$$\begin{aligned}\lim_{\substack{x \to 2p \\ x > 2p}} G(x) = G_p(2p) &= F_{2p} \quad \text{(by Theorem 4)} \\ &= G(2p).\end{aligned}$$

Finally, we see that Halsey's function is a continuous piecewise polynomial. For instance,

$$G(x) = 1, \qquad 0 < x \le 2,$$

$$G(x) = x - 1, \qquad 2 < x \le 4,$$

$$G(x) = \frac{x^2 - 5x + 10}{2}, \quad 4 < x \le 6.$$

References

1. W. G. Brady. "Addition to the Summation of Reciprocal Fibonacci and Lucas Series." *Fibonacci Quarterly 9.4* (1971):402-04.
2. H. W. Gould. "A Rearrangement of Series Based on a Partition of the Natural numbers." *Fibonacci Quarterly 15.1* (1977):66-77.
3. E. Halsey. "The Fibonacci Number F_u Where u Is Not an Integer." *Fibonacci Quarterly 3.2* (1965):147-52.
4. A. F. Horadam & A. G. Shannon. "Fibonacci and Lucas Curves." *Fibonacci Quarterly 26.1* (1988):3-13.
5. E. Lucas. "Theorie des fonctions numériques simplement périodiques." *Amer. J. Math 1* (1878):184-240.
6. F. D. Parker. "A Fibonacci Function." *Fibonacci Quarterly 6.1* (1968):1-2.

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