q-DETERMINANTS AND PERMUTATIONS

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1. Permutations

We write a *permutation* p of $\{1, 2, \ldots, n\}$ in the form $p(1)p(2)\ldots p(n)$. An *inversion* of the permutation $p(1)p(2)\ldots p(n)$ is a pair $(p(i), p(j))$ such that $p(i) > p(j)$ and $i < j$. We let $i(p)$ denote the number of inversions of p. For example, there are four inversions in the permutation $p = 2431$: $(2, 1)$, $(3, 1)$, $(4, 1)$, $(4, 3)$; hence, $i(p) = 4$.

For applications to other areas (computer science, chemistry, physics), it is useful to note that the number of inversions of the permutation $p(1)p(2)\ldots p(n)$ is the same as the minimum number of interchanges of adjacent numbers required to restore $p(1)p(2)\ldots p(n)$ to its natural order $1\,2\ldots n$.

2. Definitions

Let K be a field of characteristic 0, $K[q]$ the polynomial ring, and R a commutative ring with identity containing $K[q]$. Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in R. The ordinary determinant of A is given by the familiar formula [3, p. 14]

$$\det(A) = \sum (-1)^{i(p)} a_{1p(1)} a_{2p(2)} \cdots a_{np(n)},$$

where the summation is extended over all permutations p, and $i(p)$ is the number of inversions of the permutation p. The *q-determinant of A* is defined by the same expression with (-1) replaced by the indeterminate q:

$$\det_q(A) = \sum q^{i(p)} a_{1p(1)} a_{2p(2)} \cdots a_{np(n)}.$$

This makes q a marker for the number of inversions of a permutation.

Now, just as one can approach the subject of determinants from the point of view of Grassmann algebras, we can approach the subject of q-determinants from the point of view of q-Grassmann algebras. A *q-Grassmann algebra* (cf. [6]) is the associative $K[q]$-algebra generated by $x_1, x_2, \ldots, x_n$, satisfying the relations $x_i^2 = 0$ and $x_j x_i = q x_i x_j$, if $i < j$. Clearly, in this algebra, every monomial can be written in the *normal form*

$$c x_{i_1} x_{i_2} \ldots x_{i_r}$$

where c is in $K[q]$ and $i_1 < i_2 < \cdots < i_r$. Hence, in normal form we have

$$(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n)(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n)$$
$$\cdots (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n) = \det_q(a_{ij}) x_1 x_2 x_3 \ldots x_n.$$

3. Properties

Theorem 1:

(1) The q-determinant is a multilinear function of the rows and columns.
(2) The q-determinant of a block triangular matrix is the product of the q-determinants of the diagonal blocks.
(3) $\det_q(A) = \det_q(A^T)$, where A^T is the transpose of A.
(4) (Expansion Theorem) Let A_{ij} denote the (i, j)-minor of A. Then,

$$\begin{aligned}\det_q(A) &= a_{11}\det_q(A_{11}) + qa_{21}\det_q(A_{21}) + q^2a_{31}\det_q(A_{31}) + \cdots \\ &\qquad + q^{n-1}a_{n1}\det_q(A_{n1}) \\ &= a_{nn}\det_q(A_{nn}) + qa_{(n-1)n}\det_q(A_{(n-1)n}) + \cdots + q^{n-1}a_{1n}\det_q(A_{1n}) \\ &= a_{11}\det_q(A_{11}) + qa_{12}\det_q(A_{12}) + q^2a_{13}\det_q(A_{13}) + \cdots \\ &\qquad + q^{n-1}a_{1n}\det_q(A_{1n}) \\ &= a_{nn}\det_q(A_{nn}) + qa_{n(n-1)}\det_q(A_{n(n-1)}) + \cdots + q^{n-1}a_{n1}\det_q(A_{n1}).\end{aligned}$$

Proof: Parts (1) and (2) are obvious; (3) follows from $i(p) = i(p^{-1})$. The four equalities in (4) represent four ways of sorting the terms of $\det_q(A)$. They follow from the q-Grassmann algebra formulation of the q-determinants. [The last two equalities also follow from the first two and part (3).] Q.E.D.

4. Fibonacci Polynomials

There are several related polynomial sequences all named Fibonacci polynomials. Here by Fibonacci polynomials we mean the polynomials Riordan called $L_n(x)$ in his book [4, pp. 182-83]. They were later reintroduced by Doman and Williams in [1]. It is interesting to note that Doman and Williams were led to the definition of these polynomials from a study of a one-dimensional Ising chain in physics.

Fibonacci polynomials $F_n(q)$ are defined by the recurrence relation

$$F_{n+1}(q) = F_n(q) + qF_{n-1}(q),$$

and the initial conditions $F_0(q) = 0$, $F_1(q) = 1$. They are, in fact, expressible as

$$F_{n+1}(q) = \sum_{i=0}^{h}\binom{n-i}{i}q^i,$$

where h is the integer part of $n/2$ (for $n > 0$). As we shall show in the following, there are also the generating functions of the number of inversions of permutations p satisfying $|i - p(i)| < 2$, for all i.

5. Generating Functions

In this section, we derive several generating functions of the number of inversions of permutations by applying q-determinants to (0, 1)-matrices. We let K be the rational field, and we use the abbreviations:

$$[n] = (1 + q + q^2 + \cdots + q^{n-1}),$$

$$[n]! = [1][2][3]\ldots[n].$$

Theorem 2: The generating functions of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ is $[n]!$ ([5, p. 21]).

Proof: Let J_n denote the $n \times n$ matrix whose every entry is equal to 1. By the Expansion Theorem,

$$\sum q^{i(p)} = \det_q(J_n) = (1 + q + q^2 + \cdots + q^{n-1})\det_q(J_{n-1}) = [n]!$$

Here the summation is taken over all permutations. Q.E.D.

Theorem 3: The generating functions of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ satisfying $(i - p(i)) < r$, for all i, where $r \le n$, is $[r]^{n-r}[r]!$.

Proof: Let $K_n(r) = (k_{ij})$ denote the $n \times n$ matrix defined by

$$k_{ij} = \begin{cases} 1, \text{ if } i - j < r, \\ 0, \text{ otherwise.} \end{cases}$$

Again, by the Expansion Theorem,

$$\sum_{i - p(i) < r} q^{i(p)} = \det_q(K_n(r)) = (1 + q + q^2 + \cdots + q^{r-1})\det_q(K_{n-1}(r))$$
$$= [r]^{n-r}\det_q(K_r(r)) = [r]^{n-r}[r]! \quad \text{Q.E.D.}$$

Theorem 4: The generating functions of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ satisfying $|i - p(i)| < 2$, for all i, is the Fibonacci polynomial $F_{n+1}(q)$.

Proof: Let $L_n = (f_{ij})$ denote the $n \times n$ matrix defined by

$$f_{ij} = \begin{cases} 1, \text{ if } |i - j| < 2, \\ 0, \text{ otherwise.} \end{cases}$$

The desired generating function is then

$$\sum_{|i - p(i)| < 2} q^{i(p)} = \det_q(L_n).$$

By the Expansion Theorem, $\det_q(L_n)$ satisfies the recurrence

$$\det_q(L_{n+1}) = \det_q(L_n) + q\det_q(L_{n-1}),$$

and the initial conditions $\det_q(L_1) = 1$, $\det_q(L_2) = 1 + q$. Hence, the generating function is $F_{n+1}(q)$. Q.E.D.

We note that, since $F_{n+1}(1) = F_{n+1}$ is the Fibonacci number, the number of permutations satisfying $|i - p(i)| \le 1$ is F_{n+1} (see Example 4.7.7 of [5] and the related references given there).

Now, call $A \le B$, if $A = (a_{ij})$, $B = (b_{ij})$ are matrices with rational entries and $a_{ij} \le b_{ij}$ for all i, j. Similarly, define $f(q) \le g(q)$, if $f(q)$, $g(q)$ are polynomials with rational coefficients and the coefficient of every term q^i in $f(q)$ is less than or equal to the coefficient of the corresponding term q^i in $g(q)$. It is easy to see that if A and B are (0, 1)-matrices and $A \le B$, then $\det(A) \le \det_q(B)$ and, therefore, $0 \le \det_q(A) - \det_q(B)$.

Corollary 1: The generating function of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ such that $i - p(i) \ge r$ for some i is given by

$$[n]! - [r]^{n-r}[r]!$$

When $r = 2$, the generating function is

$$[n]! - [2]^{n-1} = [n]! - (1 + q)^{n-1},$$

and when $r = n - 1$, it is

$$[n]! - [n - 1][n - 1]! = q^{n-1}[n - 1]!$$

which is obvious from the given condition.

Corollary 2: The generating function of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ such that $|i - p(i)| \geq 2$ for some i is given by

$$[n]! - F_{n+1}(q).$$

Corollary 3: Let r be ≥ 2. The generating function of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ such that $(i - p(i)) < r$ for all i and $|i - p(i)| \geq 2$ for some i is given by

$$[r]^{n-r}[r]! - F_{n+1}(q).$$

The special case $r = 2$ of Corollary 3 is of particular interest. It says the generating function of the number of inversions of permutations of $\{1, 2, \ldots, n\}$ such that $(i - p(i)) < 2$ for all r and $|i - p(i)| \geq 2$ for some i is given by

$$(1 + q)^{n-1} - F_{n+1}(q) = \sum_{i=0}^{n-1}\left\{\binom{n-1}{i} - \binom{n-i}{i}\right\}q^i,$$

where it is understood that $\binom{r}{i} = 0$ if $r < i$.

6. Remarks

From a preprint ("Quantum Deformation of Flag Schemes and Grassmann Schemes I: A q-Deformation of the Shape-Algebra for $GL(n)$" by Earl Taft & Jacob Towber) which we received from Professor Earl Taft recently, we learned that another q-analogue of determinant (essentially replacing q by $-q^{-1}$) has been developed by Yu I. Manin.

We should also point out that the evaluation of a q-determinant is in general difficult, for the evaluation of even one of its specializations ($q = 1$), the permanent, is difficult (see [2]).

References

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