RECIPROCAL GCD MATRICES AND LCM MATRICES

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1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers. The $n \times n$ matrix $[S] = (s_{ij})$, where $s_{ij} = (x_i, x_j)$, the greatest common divisor of x_i and x_j, is called the greatest common divisor (GCD) matrix on S. The study of GCD matrices was initiated in [1]. In that paper, the authors obtained a structure theorem for GCD matrices and showed that each is positive definite, and hence nonsingular. A corollary of these results yielded a proof that, if S is factor-closed, then the determinant of S, $\det[S]$, is equal to $\phi(x_1)\phi(x_2)\ldots\phi(x_n)$, where $\phi(x)$ is Euler's totient. The set S is said to be factor-closed (FC) if all positive factors of any member of S belong to S.

In [4], Z. Li used the structure in [1] to compute a formula for the determinant of an arbitrary GCD matrix.

In this paper, we define a natural analog of the GCD matrix on S. Let $[[S]] = (t_{ij})$ be the $n \times n$ matrix with $t_{ij} = [x_i, x_j]$, the least common multiple of x_i and x_j. We shall obtain a structure theorem for $[[S]]$ and show that it is nonsingular, but never positive definite. As it turns out, the matrix factorization of $[[S]]$ emerges from the structure of the related *reciprocal GCD matrix* $1/[S]$, the i, j-entry of which is $1/(x_i, x_j)$. Reciprocal GCD matrices are addressed in the next section.

2. Reciprocal GCD Matrices

Definition 1: Let $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers. The matrix $1/[S]$ is the $n \times n$ matrix whose i, j-entry is $1/(x_i, x_j)$. We call $1/[S]$ the reciprocal GCD matrix on S.

Clearly reciprocal GCD matrices are symmetric. Furthermore, rearrangements of the elements of S yield similar matrices. Hence, as in [1] and [2], we may always assume $x_1 < x_2 < \ldots < x_n$.

We shall show that each reciprocal GCD matrix can be written as a product of A and A^T, the transpose of A, for some matrix A with complex number entries.

In what follows, we let $\mu(n)$ denote the Moebius function

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1p_2\ldots p_r, \text{ distinct prime factors} \\ 0 & \text{otherwise.} \end{cases}$$

The lower-case letter "p" will always denote a positive prime.

Definition 2: If n is a positive integer, we denote by $g(n)$ the sum

$$g(n) = \frac{1}{n} \cdot \sum_{e|n} e \cdot \mu(e).$$

We observe that $g(n) = f(n)h(n)$, where $f(n) = 1/n$ and $h(n) = \sum_{e|n} e \cdot \mu(e)$. Since f and h are multiplicative functions, g is multiplicative. Furthermore, if p is a prime, $h(p^m) = 1 - p$. Hence, $g(p^m) = (1 - p)/p^m$. It follows that

$$g(n) = \frac{1}{n} \prod_{p|n} (1 - p) = \frac{\phi(n)}{n^2} \prod_{p|n} (-p).$$

Moreover, by the Moebius Inversion Formula (see, e.g., [5]), it is true that

$$f(n) = 1/n = \sum_{e|n} g(e).$$

These results are summarized in the following lemma.

Lemma 1: Let n be a positive integer. Then $g(n) = 1$ if $n = 1$, and

$$g(n) = \frac{1}{n} \prod_{p|n} (1 - p) \text{ if } n > 1.$$

Moreover,

$$1/n = \sum_{e|n} g(e). \quad \square$$

It is clear that any set of positive integers is contained in an (minimal) FC set. We obtain the following structure theorem for reciprocal GCD matrices.

Theorem 1: Let $S = \{x_1, x_2, \ldots, x_n\}$ be ordered by $x_1 < x_2 < \ldots < x_n$. Then the reciprocal GCD matrix $1/[S]$ is the product of an $n \times m$ complex matrix A and the $m \times n$ matrix A^T, where the nonzero entries of A are of the form $\sqrt{g(d)}$ for some d in an FC set that contains S.

Proof: Suppose $F = \{d_1, d_2, \ldots, d_m\}$ is an FC set containing S. Let the complex matrix $A = (a_{ij})$ be defined as follows:

$$a_{ij} = \begin{cases} \sqrt{g(d_j)} & \text{if } d_j \text{ divides } x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(AA^T)_{ij} = \sum_{k=1}^{m} a_{ik} a_{jk} = \sum_{\substack{d_k|x_i \\ d_k|x_j}} \sqrt{g(d_k)} \cdot \sqrt{g(d_k)} = \sum_{d_k|(x_i,\, x_j)} g(d_k) = \frac{1}{(x_i,\, x_j)},$$

since F is factor-closed. Thus, $1/[S] = AA^T$. $\square$

Remark 1: Some of the entries $\sqrt{g(d_j)}$ of A in Theorem 1 may be imaginary complex numbers. A real matrix factorization for $1/[S]$ could be obtained by defining $B = (b_{ij})$ via

$$b_{ij} = \begin{cases} g(d_j) & \text{if } d_j \text{ divides } x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, if C is the incidence matrix corresponding to B, it is true that $1/[S] = B \cdot C^T$.

Corollary 1: Let S be an FC set. Then

$$\det(1/[S]) = g(x_1)g(x_2) \ldots g(x_n).$$

Proof: In Theorem 1, take $F = S$; then A and A^T are lower triangular and upper triangular, respectively. So

$$\det(1/[S]) = \det(A) \cdot \det(A^T)$$
$$= (\det(A))^2 = g(x_1)g(x_2) \ldots g(x_n). \quad \square$$

Remark 2: The set F in Theorem 1 may be chosen so that $d_1 = x_1$, $d_2 = x_2$, ..., $d_n = x_n$. Hence $A = [A_1, A_2]$, where A_1 is an $n \times n$ lower triangular matrix of the form

$$\begin{bmatrix} \sqrt{g(x_1)} & & & 0 \\ & \sqrt{g(x_2)} & & \\ & & \ddots & \\ * & & & \sqrt{g(x_n)} \end{bmatrix}.$$

Therefore, rank $(A) = n$. However, since A has nonreal entries, we cannot conclude that AA^T is nonsingular.

Remark 3: Unlike GCD matrices, reciprocal GCD matrices are *never* positive definite. Recall that the AA^T factorization in Theorem 1 is a complex matrix product, whereas, in [1], A is real. The fact that a reciprocal GCD matrix is not positive definite follows readily from the observation that its leading principal 2×2 minor

$$\frac{1}{x_1 x_2} - \frac{1}{(x_1, x_2)^2}$$

is negative.

Remark 4: As in [4], a sum formula for the determinant of an arbitrary reciprocal GCD matrix may be obtained from the Cauchy-Binet Formula (see, e.g., [3]) and the factorization AA^T. We omit this formula due to its length.

3. LCM Matrices

Definition 3: Let $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers. The $n \times n$ matrix $[[S]] = (t_{ij})$, where $t_{ij} = [x_i, x_j]$, the least common multiple of x_i and x_j, is called the least common multiple [LCM] matrix on S.

The structure and determinants of LCM matrices come directly from results on reciprocal GCD matrices, since

$$[x_i, x_j] = \frac{x_i x_j}{(x_i, x_j)}.$$

If $[[S]]$ is an LCM matrix, we may factor out x_i from Row i and x_j from Column j to obtain $1/[S]$. Hence, every LCM matrix results from performing elementary row and column operations on the corresponding reciprocal GCD matrix.

The following theorem is a direct consequence of the preceding remarks.

Theorem 2: Let $S = \{x_1, x_2, \ldots, x_n\}$ be ordered by $x_1 < x_2 < \cdots < x_n$, and let A be the $n \times n$ matrix in Theorem 1. Then

$$[[S]] = D \cdot AA^T \cdot D = D \cdot (1/[S]) \cdot D,$$

where D is the $n \times n$ diagonal matrix $\mathrm{diag}(x_1, x_2, \ldots, x_n)$. □

Corollary 2: An LCM matrix is not positive definite. □

Corollary 3: If S is an FC set, then

$$\det[[S]] = x_1^2 \ldots x_n^2 \cdot g(x_1) \ldots g(x_n) = \prod_{i=1}^{n} \left[\phi(x_i) \cdot \prod_{p \mid x_i} (-p) \right]. \quad \square$$

As before, the Cauchy-Binet formula may be used to obtain a sum formula for $\det[[S]]$, S arbitrary.

Remark 5: We know from Corollary 3 that $\det[[S]] \neq 0$ when S is FC. A natural question arises: When is $\det[[S]]$ zero? For instance, when $S = \{1, 2, 15, 42\}$, $\det[[S]] = 0$. Furthermore, when is $\det[[S]]$ positive? This does not depend entirely upon the parity of n, even in the factor-closed case. For example, when $S = \{1, 2, 4, 8\}$, $\det[[S]] < 0$, but when $S = \{1, 2, 3, 6\}$, $\det[[S]] > 0$. In view of these comments, we leave the following as a problem.

Problem: For which sets S is $\det[[S]]$ positive? For which FC sets S is $\det[[S]]$ positive? For which sets S is $\det[[S]] = 0$?

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