SECOND-ORDER RECURRENCES AND THE SCHRÖDER-BERNSTEIN THEOREM

Calvin Long
Washington State University, Pullman, WA 99164

John Bradshaw
Cariboo College, Kamloops, BC V2B 5J3, Canada
(Submitted August 1989)

1. Introduction

The Schröder-Bernstein theorem states that if f is a one-to-one mapping of X into Y and g is a one-to-one mapping of Y into X, then there exists a one-to-one mapping h of X onto Y; see, for example, [1].

The proof of the theorem involves the construction of three disjoint subsets of X satisfying certain criteria. Applied to a specific example, the subsets produced are unions of intervals bounded by ratios of successive Fibonacci and Lucas numbers and the singleton $\{2/(1 + \sqrt{5})\}$ where $(1 + \sqrt{5})/2$ is the golden ratio. More generally, the subsets produced are the unions of intervals bounded by ratios of successive elements of two general second-order recurrence sequences with the same characteristic equation and the singleton $\{1/\alpha\}$ where α denotes the positive root of the characteristic equation of the given recurrence.

As usual, we define the Fibonacci and Lucas sequences for all n by

(1) $\quad F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n$

and

(2) $\quad L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n.$

We further define the sequences $\{H_n\}$ and $\{K_n\}$ for all n by

(3) $\quad H_0 = c,\ H_1 = ac,\ H_{n+2} = aH_{n+1} + bH_n$

and

(4) $\quad K_0 = d,\ K_1 = e,\ K_{n+2} = aK_{n+1} + bK_n$

where a, b, c, d, and e are positive. Since we will need it later, we further require that

(5) $\quad a > \dfrac{e}{d}.$

2. Proof of the Schröder-Bernstein Theorem

Before showing how it is related to second-order recurrences, we outline the proof of the Schröder-Bernstein theorem.

With f and g as defined, let $g(Y)$ be the subset of X that is the image of Y under g. Let $A_0 = X - g(Y)$ and let $A_n = g(f(A_{n-1}))$ for each integer $n \geq 1$. Let $f(X)$ be the subset of Y that is the image of X under f, let $B_0 = g(Y - f(X))$, and let $B_n = g(f(B_{n-1}))$ for all $n \geq 1$. Finally, set

$$A = \bigcup_{i=0}^{\infty} A_i, \quad B = \bigcup_{i=0}^{\infty} B_i, \quad \text{and} \quad X_\infty = X - (A \cup B).$$

Then it is not difficult to show that A, B, and X_∞ are disjoint, that

$$X = A \cup B \cup X_\infty,$$

and that the function h, defined by

$$h(x) = \begin{cases} f(x) & \text{for } x \in A \cup X_\infty, \\ g^{-1}(x) & \text{for } x \in B, \end{cases}$$

is a one-to-one mapping from X onto Y.

3. An Example Involving Second-Order Recurrences

Theorem 1: Let $X = (0, 2)$, $Y = (1, \infty)$, $f(x) = x + 1$, and $g(y) = 1/y$. Then the sets A_n, B_n, and X_∞ of the proof of the Schröder-Bernstein theorem are given by the following:

$$A_n = \begin{cases} \left[\dfrac{F_{n+1}}{F_{n+2}}, \dfrac{L_n}{L_{n+1}}\right) & n \geq 0,\ n \text{ even}, \\ \left(\dfrac{L_n}{L_{n+1}}, \dfrac{F_{n+1}}{F_{n+2}}\right] & n \geq 0,\ n \text{ odd}, \end{cases} \tag{6}$$

$$B_n = \begin{cases} \left(\dfrac{F_n}{F_{n+1}}, \dfrac{L_{n+1}}{L_{n+2}}\right] & n \geq 0,\ n \text{ even}, \\ \left[\dfrac{L_{n+1}}{L_{n+2}}, \dfrac{F_n}{F_{n+1}}\right) & n \geq 0,\ n \text{ odd}, \end{cases} \tag{7}$$

and

$$X_\infty = \left\{\frac{2}{1 + \sqrt{5}}\right\}. \tag{8}$$

More generally, the following theorem holds.

Theorem 2: Let a, b, c, d, e, $\{H_n\}_{n \geq 0}$, and $\{K_n\}_{n \geq 0}$ be as in the introduction. Let $X = (0, d/e)$, $Y = (a, \infty)$, $f(x) = bx + a$, $g(y) = 1/y$, and $H_{-1} = 0$. Then the sets A_n, B_n, and X_∞ of the proof of the Schroder-Bernstein theorem are given by

$$A_n = \begin{cases} \left[\dfrac{H_n}{H_{n+1}}, \dfrac{K_n}{K_{n+1}}\right) & n \geq 0,\ n \text{ even}, \\ \left(\dfrac{K_n}{K_{n+1}}, \dfrac{H_n}{H_{n+1}}\right] & n \geq 0,\ n \text{ odd}, \end{cases} \tag{9}$$

$$B_n = \begin{cases} \left(\dfrac{H_{n-1}}{H_n}, \dfrac{K_{n+1}}{K_{n+2}}\right] & n \geq 0,\ n \text{ even}, \\ \left[\dfrac{K_{n+1}}{K_{n+2}}, \dfrac{H_{n-1}}{H_n}\right) & n \geq 0,\ n \text{ odd}, \end{cases} \tag{10}$$

and

$$X_\infty = \left\{\frac{2}{a + \sqrt{a^2 + 4b}}\right\}. \tag{11}$$

Before proving these theorems, it will be convenient to prove the following lemmas.

Lemma 1: If f, g, $\{H_n\}$, and $\{K_n\}$ are as defined in Theorem 2, then

$$g\left(f\left(\frac{H_{n-1}}{H_n}\right)\right) = \frac{H_n}{H_{n+1}} \quad \text{and} \quad g\left(f\left(\frac{K_n}{K_{n+1}}\right)\right) = \frac{K_{n+1}}{K_{n+2}}$$

for all $n \geq 0$.

Proof: Since $g(f(x)) = 1/(bx + a)$,

$$g\left(f\left(\frac{H_{n-1}}{H_n}\right)\right) = \frac{1}{b \cdot \dfrac{H_{n-1}}{H_n} + a} = \frac{H_n}{bH_{n-1} + aH_n} = \frac{H_n}{H_{n+1}}$$

by (3). Note that this is even true for $n = 0$, since $H_{-1} = 0$ is consistent with (3). Similarly, we show that

$$g\left(f\left(\frac{K_n}{K_{n+1}}\right)\right) = \frac{K_{n+1}}{K_{n+2}}.$$

Lemma 2: For the sequences $\{H_n\}$ and $\{K_n\}$ as defined in Theorem 2, the following inequalities hold.

(12) $\qquad \dfrac{H_n}{H_{n+1}} > \dfrac{K_{n+1}}{K_{n+2}} \qquad n \geq 0,\ n$ even,

(13) $\qquad \dfrac{H_n}{H_{n+1}} < \dfrac{K_{n+1}}{K_{n+2}} \qquad n \geq 0,\ n$ odd.

Proof: Since a, b, c, d, and e are positive, it follows from (3) and (4) that

$$\frac{H_0}{H_1} = \frac{c}{ac} > \frac{e}{ae + bd} = \frac{K_1}{K_2}.$$

Since $g(f(x)) = 1/(bx + a)$ is a decreasing function and

$$f\left(\frac{H_0}{H_1}\right) = \frac{H_2}{H_1} \quad \text{and} \quad f\left(\frac{K_1}{K_2}\right) = \frac{K_3}{K_2},$$

it follows that

$$\frac{H_1}{H_2} < \frac{K_2}{K_3}$$

and the argument for all $n \geq 0$ is easily completed by induction.

Lemma 3: If X, A_n, and B_n are defined as in Theorem 2, then

$$X - \left[\left(\bigcup_{i=0}^{n} A_i\right) \cup \left(\bigcup_{i=0}^{n} B_i\right)\right] = \begin{cases} \left(\dfrac{K_{n+1}}{K_{n+2}}, \dfrac{H_n}{H_{n+1}}\right) & n \geq 0,\ n \text{ even}, \\[2ex] \left(\dfrac{H_n}{H_{n+1}}, \dfrac{K_{n+1}}{K_{n+2}}\right) & n \geq 0,\ n \text{ odd}. \end{cases}$$

Proof: For $n = 0$, it follows from (3) and (4) that

$$A_0 \cup B_0 = \left[\frac{H_0}{H_1}, \frac{K_0}{K_1}\right) \cup \left(\frac{H_{-1}}{H_0}, \frac{K_1}{K_2}\right] = \left[\frac{c}{ac}, \frac{d}{e}\right) \cup \left(\frac{0}{c}, \frac{e}{ae + bd}\right].$$

Thus, since $X = (0, d/e)$ and $e/(ae + bd) < c/ac$ as above,

$$X - (A_0 \cup B_0) = \left(\frac{e}{ae + bd}, \frac{c}{ac}\right) = \left(\frac{K_1}{K_2}, \frac{H_0}{H_1}\right)$$

as claimed. Assume that, for k even,

$$X - \left[\left(\bigcup_{i=0}^{k} A_i\right) \cup \left(\bigcup_{i=0}^{k} B_i\right)\right] = \left(\frac{K_{k+1}}{K_{k+2}}, \frac{H_k}{H_{k+1}}\right).$$

Then, since $H_{k+1}/H_{k+2} < K_{k+2}/K_{k+3}$ by Lemma 2, it follows that

$$X - \left[\left(\bigcup_{i=0}^{k+1} A_i\right) \cup \left(\bigcup_{i=0}^{k+1} B_i\right)\right] = \left(\frac{K_{k+1}}{K_{k+2}}, \frac{H_k}{H_{k+1}}\right) - \left(\frac{K_{k+1}}{K_{k+2}}, \frac{H_{k+1}}{H_{k+2}}\right] - \left[\frac{K_{k+2}}{K_{k+3}}, \frac{H_k}{H_{k+1}}\right) =$$

$$= \left(\frac{H_{k+1}}{H_{k+2}}, \frac{K_{k+2}}{K_{k+3}}\right).$$

This proves the result for $k + 1$. The proof for $k + 2$, which completes the induction, is the same as for $k + 1$ except that it requires the inequality

$$\frac{K_{k+3}}{K_{k+4}} < \frac{H_{k+2}}{H_{k+3}}$$

which also follows from Lemma 2, since k is even.

Proof of Theorem 2: As in the sketch of the proof of the Schröder-Bernstein theorem, we consider

$$g(Y) = g((a, \infty)) = \left(0, \frac{1}{a}\right)$$

and

$$A_0 = X - g(Y) = \left(0, \frac{d}{e}\right) - \left(0, \frac{1}{a}\right) = \left[\frac{1}{a}, \frac{d}{e}\right) = \left[\frac{H_0}{H_1}, \frac{K_0}{K_1}\right)$$

by (3) and (4), since $1/a < d/e$ by (5). Now, assume that

$$A_k = \left[\frac{H_k}{H_{k+1}}, \frac{K_k}{K_{k+1}}\right),$$

where $k \geq 0$ is even. Then

$$A_{k+1} = g(f(A_k)) = g\left(f\left(\left[\frac{H_k}{H_{k+1}}, \frac{K_k}{K_{k+1}}\right)\right)\right) = \left(\frac{K_{k+1}}{K_{k+2}}, \frac{H_{k+1}}{H_{k+2}}\right]$$

by Lemma 1 since, as noted above, $g(f(x))$ is a decreasing function. Repeating this argument with $k + 1$ replacing k, we have that

$$A_{k+2} = \left[\frac{H_{k+2}}{H_{k+3}}, \frac{K_{k+2}}{K_{k+3}}\right).$$

Thus, by mathematical induction, the A_n are as described in (9). Moreover, we note that we have also shown that

$$\frac{H_n}{H_{n+1}} < \frac{K_n}{K_{n+1}} \text{ for } n \text{ even} \quad \text{and} \quad \frac{H_n}{H_{n+1}} > \frac{K_n}{K_{n+1}} \text{ for } n \text{ odd}.$$

To prove (10), we recall from the sketch of the Schröder-Bernstein theorem that

$$B_0 = g(Y - f(X)) = g\left((a, \infty) - f\left(\left(0, \frac{d}{e}\right)\right)\right) = g\left((a, \infty) - \left(a, \frac{bd + ae}{e}\right)\right)$$

$$= g\left(\left[\frac{bd + ae}{e}, \infty\right)\right) = \left(0, \frac{e}{bd + ae}\right] = \left(\frac{H_{-1}}{H_0}, \frac{K_1}{K_2}\right],$$

since we take $H_{-1} = 0$ as noted in the proof of Lemma 1. The proof of (10) is now completed by induction exactly like the proof of (9). Finally, to prove (11), we use Lemma 3. As in the sketch of the Schröder-Bernstein theorem

$$X_\infty = X - (A \cup B)$$

$$= X - \left[\left(\bigcup_{i=0}^{\infty} A_i\right) \cup \left(\bigcup_{i=0}^{\infty} B_i\right)\right] = \lim_{n\to\infty}\left\{X - \left[\left(\bigcup_{i=0}^{n} A_i\right) \cup \left(\bigcup_{i=0}^{n} B_i\right)\right]\right\}$$

$$= \lim_{n\to\infty} \begin{cases} \left(\dfrac{K_{n+1}}{K_{n+2}}, \dfrac{H_n}{H_{n+1}}\right) & n \text{ even}, \\ \left(\dfrac{H_n}{H_{n+1}}, \dfrac{K_{n+1}}{K_{n+2}}\right) & n \text{ odd}, \end{cases} = \{1/\alpha\},$$

where $\alpha = (a + \sqrt{a^2 + 4b})/2$, since it is well known that

$$\lim_{n\to\infty} \frac{K_{n+1}}{K_n} = \lim_{n\to\infty} \frac{H_{n+1}}{H_n} = \alpha,$$

the positive root of the characteristic equation of the recurrences in (3) and (4). This completes the proof of Theorem 2.

It is interesting to note that, since α is a root of $x^2 = ax + b$,

$$f(g(\alpha)) = a + \frac{b}{\alpha} = \frac{a\alpha + b}{\alpha} = \frac{\alpha^2}{\alpha} = \alpha$$

and

$$g\left(f\left(\frac{1}{\alpha}\right)\right) = \frac{1}{(b/\alpha) + a} = \frac{\alpha}{b + a\alpha} = \frac{\alpha}{\alpha^2} = \frac{1}{\alpha}$$

so that α is a fixed point of $f(g(x))$ and $1/\alpha$ is a fixed point of $g(f(x))$.

Proof of Theorem 1: This follows immediately from Theorem 2 by taking $a = b = c = e = 1$ and $d = 2$.

Of course, similar results obtain for the Pell and other well-known sequences by other appropriate choices of a, b, c, d, and e.

Reference

1. George F. Simmons. *Introduction to Topology and Modern Analysis*. New York: McGraw-Hill, 1963, pp. 29-30.

*****