SOME RECURSIVE ASYMPTOTES

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1. Introduction

We consider a generalized Fibonacci sequence $\{H_n\}$ defined by the linear homogeneous recurrence relation

$$(1.1) \qquad H_n = H_{n-1} + H_{n-2},\ n > 2,$$

with initial conditions $H_1 = 1$, $H_2 = X$, where X can be real or complex. In [6] Horadam has studied the properties of these sequences. Among these properties, it is well known that

$$(1.2) \qquad \lim_{n\to\infty} H_{n+1}/H_n = \alpha,$$

where $\alpha = (1 + \sqrt{5})/2$ is the positive root of the associated auxiliary polynomial. The other root is $\beta = (1 - \sqrt{5})/2$.

The purpose of this paper is to look at two variations of (1.2) and at two curves that result. Before that, we recall that the general term for $\{H_n\}$ is given by

$$(1.3) \qquad H_n = A\alpha^n - B\beta^n,$$

where $A = (X - \beta)/\alpha\sqrt{5}$ and $B = (X - \alpha)/\beta\sqrt{5}$.

2. Curves

We next construct the function

$$(2.1) \qquad I(X) = \lim_{n\to\infty} \alpha^{2n}\left|\alpha - \frac{H_{n+1}}{H_n}\right|.$$

At first sight, this would appear to be indeterminate. However, with the use of (1.3), we can establish that

$$(2.2) \qquad I(X) = \pm\frac{\alpha - \beta}{A/B}$$

accordingly as n is even or odd. With the repeated use of $\alpha^2 = \alpha + 1$, (2.2) can be reduced to

$$(2.3) \qquad I(x) = \pm(3\alpha + 1)(X - \alpha)/(X - \beta).$$

Figure 1 is a sketch of $I(X)$ plotted on the Cartesian plane. We have a pair of intersecting hyperbolae with asymptotes given by $I(X) = \pm(3\alpha + 1)$ and $X = \beta$, and X-intercept of $X = \alpha$.

Now put $X = x + iy$, so that we have $I(X) \equiv I(x, y)$ and

$$(2.4) \qquad I(x, y) = \pm\frac{(3\alpha + 1)}{(\alpha + (x - 1))^2 + y^2}\{(x^2 + y^2 - x - 1) + iy(2\alpha - 1)\}.$$

Figure 2 is a sketch of $I(x, y)$ plotted on the Argand plane, holding y constant and varying x. We have a pair of parabolic pencils of coaxal circles. The radius of each circle is $(5 + 5\alpha)/2y$, and each is tangential to the real axis at the points $(\pm(3\alpha + 1), 0)$.

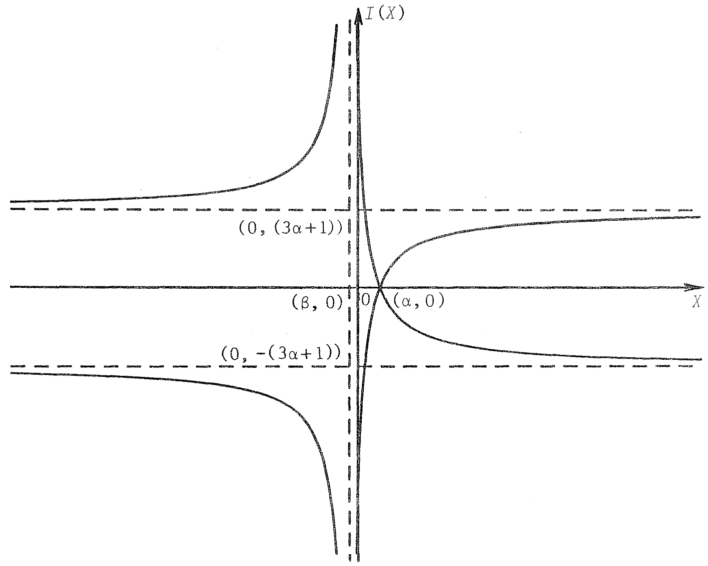

FIGURE 1. $I(X)$ vs. X

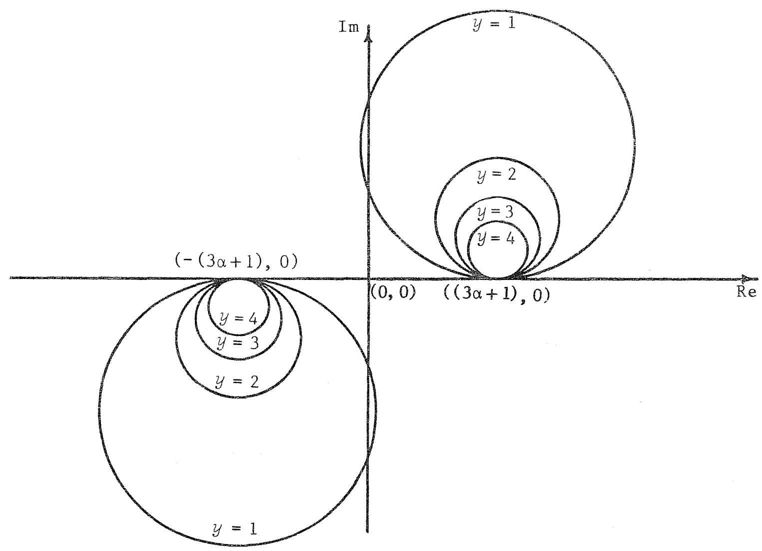

FIGURE 2. $I(x, y)$

In the terminology of Deakin [3], consider the following numbers:

complex: $x + iy$, $i^2 = -1$,
dual: $x + \varepsilon y$, $\varepsilon^2 = 0$,
duo: $x + \omega y$, $\omega^2 = 1$.

$I(x, y)$ generates circles in the complex plane, parabolas in the dual plane, and hyperbolas in the duo plane.

3. Other Generators of α

We define the iterative root sequence $\{\nu_n\} \equiv \{\nu_n(k, x; a, b)\}$ by means of the recurrence relation

(3.1) $\quad \nu_n(k, x; a, b) = (b\nu_{n-1}(k, x; a, b) + a)^{1/k}$

with initial term $\nu_1(k, x; a, b) = x^{1/k}$. For example (see [13]),

$$\lim_{n \to \infty} \nu_n(k, a; a, b) = \alpha.$$

It is known (see [2]) that if $\lim_{n \to \infty} \nu_n(k, a; a, b) = A$ then

(3.2) $\quad a + bA = A^k$

and

(3.3) $\quad \lim_{n \to \infty} \nu_n(k, F_{k-1}; F_{k-1}, F_k) = \alpha$

or

(3.4) $\quad F_{k-1} + F_k \alpha = \alpha^k,$

where F_k is the k^{th} Fibonacci number. An early example of (3.4) occurs in Basin & Hoggatt [1] and a later geometric illustration in Schoen [11]. For a background to this in the more usual context of continued fractions, see Hoggatt & Bruckman [5] and Kiss [9]. We wish to consider here the rate of convergence of (3.3).

Whitaker [12] recently showed that for sequences $\{\nu_n(k, x; a, b)\}$, a finite nonzero function $I(X)$ can be constructed in the form

$$I(X) = \lim_{n \to \infty}\left(\frac{kA^{k-1}}{b}\right)^n (A - \nu_n(k, X; a, b)).$$

The equivalent form for the Fibonacci case is

(3.5) $$I(X) = \lim_{n \to \infty}\left(\frac{k\alpha^{k-1}}{F_k}\right)^n (\alpha - \nu_n(k, X; F_{k-1}, F_k)).$$

As before, this can be considered on the real or complex planes, although there is no closed form for $I(X)$. Comparing (2.1) and (3.5), we can comment on the rates of convergence of the methods of generating α from the ratio of successive terms of the generalized Fibonacci sequence and from the iterative root sequence. The rate of convergence of the former method is proportional to α^2, whereas the other rate is proportional to $k\alpha^{k-1}/F_k$. If $k \geq 2$, $k\alpha^{k-1}/F_k > \alpha^2$, because

$$\alpha^2 F_k = (\alpha^3\alpha^{k-1} - \beta^{k-2})/\sqrt{5} < (\alpha^3/\sqrt{5})\alpha^{k-1} = (1.89)\alpha^{k-1}.$$

Thus, the iterative root sequences produce the faster convergence rate. If we consider noninteger values of k, we can find an iterative root sequence that converges to α at the same rate as the ratio of the generalized Fibonacci number; that is, we can find k, such that (i) $k\alpha^{k-1} = \alpha^2$ and (ii) $a + \alpha = \alpha^k$. This occurs when $k = 1.790048745$ and $a = 0.74841991$. Calculation shows that both H_{n+1}/H_n and $\nu_n(k, a; a, 1)$ with these values of k and a require 22 iterations to provide eight-figure accuracy for α.

4. Concluding Comments

The ideas presented here can be extended by altering the recurrence relation. One way is to include real coefficients, another is to increase the order. Kiss & Ticky [10] have determined the asymptotic distribution function for the ratios of the terms in the former case, and Goldstern, Tichy & Turnwald [4] in the latter. They have also established several estimates for the discrepancy or error term. Another generalization would be to consider

(4.1) $$I(X) = \lim_{n\to\infty} \alpha^{2n}\left|\alpha^k - \frac{H_{n+k}}{H_n}\right|$$

by analogy with (3.1) of Horadam [7]. In the Fibonacci sequence (4.1) can be rearranged as

(4.2) $$I(X)/(3\alpha + 1) = \pm F_k(X - \alpha)/(X - \beta).$$

Graphs of these are directly related to Fibonacci sequences as in Horadam & Shannon [8].

References

1. S. L. Basin & V. E. Hoggatt, Jr. "A Primer on the Fibonacci Sequence." *The Fibonacci Quarterly 1.2* (1963):61-68.
2. R. S. Beard. "Powers of the Golden Section." *The Fibonacci Quarterly 4.2* (1966):163-67.
3. M. A. B. Deakin. "Are There Parabolic Functions?" *Australian Mathematics Teacher 27.4* (1971):136-38.
4. M. Goldstern, R. F. Tichy, G. Turnwald. "Distribution of the Ratios of the Terms of a Linear Recurrence." *Monatshefte für Mathematik 107.1* (1989): 35-55.
5. V. E. Hoggatt, Jr., & P. S. Bruckman. "Periodic Continued Fraction Representations of Fibonacci-Type Irrationals." *The Fibonacci Quarterly 15.3* (1977):225-32.
6. A. F. Horadam. "A Generalised Fibonacci Sequence." *Amer. Math. Monthly 68.5* (1961):455-59.
7. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly 3.3* (1965):161-76.
8. A. F. Horadam & A. G. Shannon. "Fibonacci and Lucas Curves." *The Fibonacci Quarterly 26.1* (1988):3-13.
9. P. Kiss. "On Second Order Recurrences and Continued Fractions." *Bulletin of the Malaysian Mathematical Society (Series 2) 5.1* (1982):33-41.
10. P. Kiss & R. F. Tichy. "Distribution of the Ratios of the Terms of a Second Order Linear Recurrence." *Indagationes Mathematicae 48.1* (1986):79-86.
11. R. Schoen. "The Fibonacci Sequence in Successive Partitions of a Golden Triangle." *The Fibonacci Quarterly 20.2* (1982):159-63.
12. R. N. Whitaker. "Some Generalised Iterative Roots." *Australian Mathematical Society Gazette 15.6* (1988):149-50.
13. R. N. Whitaker. "Fun with Fibonaccie." *Function 13.3* (1989):73-79.

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