A NEW NUMERICAL TRIANGLE SHOWING LINKS WITH FIBONACCI NUMBERS

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0. Introduction

In the study of electrical networks, it is well known that the coefficients of the polynomial which characterizes the transfer function (ratio of output to input voltage) of a ladder network formed by a cascade of N identical uncoupled elementary cells belong to the $(N+1)^{\text{th}}$ row of Pascal's triangle. This circumstance allows us a direct and fast determination of the transfer function of the entire ladder network.

On the other hand, in the case of direct coupling among interacting elementary cells forming a ladder network, the polynomial coefficients are not those belonging to Pascal's triangle, but rather to another triangle named the "DFF triangle" from the initials of the authors who first dealt with it (see [3], [4]).

The DFF triangle also provides a noteworthy interest from the mathematical point of view, because some of its properties are connected with Fibonacci numbers.

1. The Generating Polynomials

The DFF triangle can be formed in the following manner ($a_{n,k}$ being the general coefficient).

We define (see [3], [4]):

(1.1) $\quad a_{n,k} = 0 \quad$ if $n < k$,

(1.2) $\quad a_{n,k} = 1 \quad$ if $n = k$, $k = 0$,

while the other elements of the triangle can be derived from the recursive formula

(1.3) $$a_{n,k} = a_{n-1,k} + \sum_{\alpha=0}^{n-1} a_{\alpha,k-1} \quad \text{if } n > k.$$

In this manner we have the DFF triangle for values of $a_{n,k}$:

$n \backslash k$	0	1	2	3	4	5	6	7 ...
0	1							
1	1	1						
2	1	3	1					
3	1	6	5	1				
4	1	10	15	7	1			
5	1	15	35	28	9	1		
6	1	21	70	84	45	11	1	
7	1	28	126	210	165	66	13	1
...	...	...	...	...	...	...	...	...

Thus, for example, $a_{3,2} = 5$ and $a_{7,5} = 66$.

The generating polynomial $P_n(x)$ is defined in [1] as

$$(1.4)\qquad P_n(x) = \sum_{k=0}^{n} \mathbf{a}_{n,k} x^k$$

where

$$(1.5)\qquad \mathbf{a}_{n,k} = \left.\frac{D^k P_n(x)}{k!}\right|_{x=0}.$$

From the DFF triangle it is possible to obtain the expression of the polynomial for small values of n:

$$(1.6)\qquad \begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 1 + x \\ P_2(x) &= 1 + 3x + x^2 \\ P_3(x) &= 1 + 6x + 5x^2 + x^3 \end{aligned}$$

and so on.

From (1.1), (1.2), (1.3), and (1.4) we have

$$(1.7)\qquad \sum_{k=0}^{n} \mathbf{a}_{n,k} x^k = \sum_{k=0}^{n} \mathbf{a}_{n-1,k} x^k + \sum_{k=0}^{n} \sum_{\alpha=0}^{n-1} \mathbf{a}_{\alpha,k-1} x^k$$

and

$$(1.8)\qquad \begin{aligned} P_n(x) &= \mathbf{a}_{n-1,n} x^n + \sum_{k=0}^{n-1} \mathbf{a}_{n-1,k} x^k + x \sum_{k=0}^{n} \sum_{\alpha=0}^{n-1} \mathbf{a}_{\alpha,k-1} x^{k-1} \\ &= P_{n-1}(x) + x \sum_{\alpha=0}^{n-1} \sum_{k=0}^{\alpha+1} \mathbf{a}_{\alpha,k-1} x^{k-1}, \end{aligned}$$

$$(1.9)\qquad P_n(x) = P_{n-1}(x) + x \sum_{\alpha=0}^{n-1} P_\alpha(x),$$

which is the recursive formula for the polynomials.

With the initial condition $P_0(x) = 1$, it is easy to obtain the polynomials (1.6). Furthermore, we can also use (1.5) to find the triangle coefficients.

In order to find the polynomials, we must apply the previous method. Let

$$(1.10)\qquad f(x, t) = \sum_{n=1}^{\infty} P_n(x) t^n.$$

Then

$$(1.11)\qquad P_n(x) = \left.\frac{D^n[f(x, t)]}{n!}\right|_{t=0}.$$

From (1.9) and (1.10) we have

$$(1.12)\qquad \begin{aligned} f(x, t) &= \sum_{n=1}^{\infty} P_{n-1}(x) t^n + x \sum_{n=1}^{\infty} \sum_{\alpha=0}^{n-1} P_\alpha(x) t^n \\ &= t \sum_{n=1}^{\infty} P_{n-1}(x) t^{n-1} + x \sum_{n=1}^{\infty} t^n [P_0 + P_1 + \cdots + P_{n-1}] \\ &= t[1 + f(x, t)] + x[1 + f(x, t)] \sum_{k=1}^{\infty} t^k = \frac{-t^2 + t(1 + x)}{t^2 - t(2 + x) + 1}. \end{aligned}$$

If we develop the denominator in (1.12) in partial fractions, we obtain

$$(1.13)\qquad f(x, t) = \frac{a(x) - 1/2}{t - b(x)/2} + \frac{-a(x) - 1/2}{t - c(x)/2} - 1,$$

where

$$y \equiv y(x) = (x^2 + 4x)^{1/2}, \quad a(x) = \frac{-y}{2(x + 4)},$$
$$b(x) = 2 + x + y, \quad \text{and} \quad c(x) = 2 + x - y.$$

From the binomial expansion in (1.13) and after simplification, we also have

$$(1.14)\quad f(x,\ t) = \frac{x+y+4}{(x+4)(x+y+2)}\sum_{n\ge 1}\left[\frac{t}{b(x)/2}\right]^n + \frac{x-y+4}{(x+4)(x-y+2)}\sum_{n\ge 1}\left[\frac{t}{c(x)/2}\right]^n = \sum_{n\ge 1}\left[\frac{1+y/(x+4)}{(x+y+2)^{n+1}/2^n} + \frac{1-y/(x+4)}{(x-y+2)^{n+1}/2^n}\right]t^n$$

from which we have, using (1.10),

$$P_n(x) = \frac{1+y/(x+4)}{(x+y+2)^{n+1}/2^n} + \frac{1-y/(x+4)}{(x-y+2)^{n+1}/2^n} = \frac{(x-y+4)(x-y+2)^n + (x+y+4)(x+y+2)^n}{(x+4)2^{n+1}},$$

$$(1.15)\quad P_n(x) = \frac{1}{2^{n+1}}\left[\frac{x-y+4}{x+4}\sum_{h=0}^{n}(-1)^h\binom{n}{h}(x+2)^{n-h}y^h + \frac{x+y+4}{x+4}\sum_{h=0}^{n}\binom{n}{h}(x+2)^{n-h}y^h\right].$$

From this equation, on distinguishing the case of odd h from that of even h, and since $y = (x^2 + 4x)^{1/2}$, we can write

$$(1.16)\quad P_n(x) = \frac{1}{2^n}\left[\sum_{h\equiv 0 \pmod 2}^{n}\binom{n}{h}(x+2)^{n-h}x^{h/2}(x+4)^{h/2} + \sum_{h\equiv 1 \pmod 2}^{n}\binom{n}{h}(x+2)^{n-h}x^{(h+1)/2}(x+4)^{(h-1)/2}\right].$$

2. Determination of $\mathbf{a}_{n,\,k}$

From equations (1.5) and (1.16), and considering also Leibniz's formula

$$(2.1)\quad D^k[f(x)g(x)] = \sum_{j=0}^{k}\binom{k}{j}D^j f(x)D^{k-j}g(x),$$

we have

$$(2.2)\quad \mathbf{a}_{n,\,k} = \frac{1}{k!\,2^n}\left[\sum_{h\equiv 0 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{k}{j}D^j[x^{h/2}(x+4)^{h/2}]\,D^{k-j}[x+2]^{n-h} + \sum_{h\equiv 1 \pmod 2}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{k}{j}D^j[x^{(h+1)/2}(x+4)^{(h-1)/2}]\,D^{k-j}[x+2]^{n-h}\right]_{x=0}$$

Then, from (2.1), it is possible to write

$$D^j[x^{h/2}(x+4)^{h/2}] = \sum_{m=0}^{j}\binom{j}{m}\binom{h/2}{m}m!\,x^{(h/2)-m} * \binom{h/2}{j-m}(j-m)!\,(x+4)^{(h/2)-j+m},$$

$$D^{k-j}[(x+2)^{n-h}] = \binom{n-h}{k-j}(k-j)!\,(x+2)^{n-h-k+j},$$

and

$$(2.3)\qquad D^j[x^{(h+1)/2}(x+4)^{(h-1)/2}] = \sum_{m=0}^{j}\binom{j}{m}\binom{(h+1)/2}{m}m!\,x^{((h+1)/2)-m} * \\ * \binom{(h-1)/2}{j-m}(j-m)!\,(x+4)^{((h-1)/2)-j+m},$$

where here and in the following equations the * represents multiplication.

From (2.3) and from the properties of binomial coefficients, (2.2) becomes

$$(2.4)\qquad \mathrm{a}_{n,k} = \frac{1}{2^n}\Bigg\{\sum_{h\equiv 0\ (\mathrm{mod}\ 2)}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{n-h}{k-j}(x+2)^{n-h-k+j} * \\ * \sum_{m=0}^{j}\binom{h/2}{m}\binom{h/2}{j-m}x^{(h/2)-m}(x+4)^{(h/2)-j+m} \\ + \sum_{h\equiv 1\ (\mathrm{mod}\ 2)}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{n-h}{k-j}(x+2)^{n-h-k+j}\sum_{m=0}^{j}\binom{(h+1)/2}{m}\binom{(h-1)/2}{j-m} * \\ * x^{((h+1)/2)-m}(x+4)^{((h-1)/2)-j+m}\Bigg\}_{x=0}.$$

When $x = 0$, the m-sum exists only if $m = h/2$ and $m = (h+1)/2$, respectively. So we can write

$$(2.5)\qquad \mathrm{a}_{n,k} = \sum_{h=0}^{n}\binom{n}{h}\sum_{j=0}^{k}\binom{n-h}{k-j}2^{h-k-j}\left[\binom{h/2}{j-h/2} + \binom{(h-1)/2}{j-(h+1)/2}\right].$$

It is worth pointing out that $\binom{a}{b} = 0$ if $b \notin \mathbb{N}_0$, so

$$\binom{h/2}{j-h/2} \neq 0 \quad \text{only if } h \text{ is even}$$

and

$$\binom{(h-1)/2}{j-(h+1)/2} \neq 0 \quad \text{only if } h \text{ is odd.}$$

3. The Properties of $\mathrm{a}_{n,k}$

3.1 The Asymptotic Expression of $\mathrm{a}_{n,k}$

From [2], the asymptotic expression of the binomial coefficient is

$$(3.1)\qquad \binom{n}{k} \simeq \left(\frac{2}{\pi n}\right)^{1/2} 2^n \exp\left(-\frac{2((n/2)-k)^2}{n}\right)$$

and, from equation (2.5), we find that the asymptotic expression of $\mathrm{a}_{n,k}$ can be expressed as

$$\mathrm{a}_{n,k} \simeq \frac{2^{2n-k+2}}{\pi^{3/2}}\sum_{h=0}^{n}\frac{1}{(n(n-h))^{1/2}}\sum_{j=0}^{k}2^{-j} * \\ * \exp\left(\frac{-2(n-h)((n/2)-h)^2 - 2n[((n-h)/2)-(k-j)]^2}{n(n-h)}\right) * \\ * \left\{\frac{2^{h/2}}{h^{1/2}}\exp\left[-\frac{4}{h}\left(\frac{-3}{4}h - j\right)^2\right]_{h\ \mathrm{even}} \right. \\ \left. + \frac{2^{(h-1)/2}}{(h-1)^{1/2}}\exp\left[-\frac{4}{h-1}\left(\frac{3h+1}{4} - j\right)^2\right]_{h\ \mathrm{odd}}\right\}.$$

3.2 The Row Sums of the Triangle Are Equal to Fibonacci Numbers with Odd Subscripts

From the expression (1.16) for $P_n(x)$, when $x = 1$, we have

$$(3.3) \qquad P_n(1) = \frac{(5 + 5^{1/2})/5}{(3 + 5^{1/2})^{n+1}/2^n} + \frac{(5 - 5^{1/2})/5}{(3 - 5^{1/2})^{n+1}/2^n}$$

$$= \frac{1}{5^{1/2}}\left[\frac{1 + 5^{1/2}}{2}\left(\frac{3 - 5^{1/2}}{2}\right)^{n+1} - \frac{1 - 5^{1/2}}{2}\left(\frac{3 + 5^{1/2}}{2}\right)^{n+1}\right].$$

From Binet's formula, we have

$$(3.4) \qquad F_{2n+1} = \frac{1}{5^{1/2}}\left[\left(\frac{1 + 5^{1/2}}{2}\right)^{2n+1} - \left(\frac{1 - 5^{1/2}}{2}\right)^{2n+1}\right].$$

It is easy to show that $P_n(1) = F_{2n+1}$ (where $F_1 = 1$, $F_3 = 2$, $F_5 = 5$, ...).

This is the main result we were interested in showing in this paper. (It may also be verified in the table of the DFF triangle.)

3.3 The Sums of the Triangle Diagonals Give the Powers of 2

From a direct inspection of the DFF triangle and (1.3), we have that the sum of the elements of an upward-slanting diagonal is equal to the sum of all elements that are above this diagonal and, consequently, to the sum of all superior upward-slanting diagonals. This sum value is a power of 2.

In fact, if we define

$$\sum^n = \sum_{r=0}^{n} a_{n-r,\, r},$$

it is possible to write

$$\sum^n = \sum^{n-1} + \sum^{n-2} + \cdots + \sum^1 + 1$$

$$= 2\left(\sum^{n-2} + \sum^{n-3} + \cdots + \sum^1 + 1\right) = \cdots = 2^{n-2}\left(\sum^1 + 1\right) = 2^{n-1},$$

since $\sum^1 = 1$.

4. Conclusions

The principal aim of this paper has been the determination of a closed expression of the general coefficient $a_{n,k}$ of a new numerical triangle, named the DFF, which characterizes the transfer function of a ladder network whose elementary cells are directly coupled. Moreover, the authors present some of the triangle's interesting mathematical properties, one of which is connected to Fibonacci numbers.

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