NOTE ON THIRD ORDER DETERMINANTS

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The recent exhaustive investigation of nine-digit determinants by Bicknell and Hoggatt that appeared in the Mathematics Magazine of May-June, 1963, raises an interesting question [1]. Given that

$$\begin{vmatrix} 9 & 4 & 2 \\ 3 & 8 & 6 \\ 5 & 1 & 7 \end{vmatrix}$$

or any equivalent arrangement producing the same set of products has a maximum value of 412, would we obtain a maximum for any other nine consecutive positive integers using the same relative arrangement? This note will offer a negative answer and indicate the maximum for all positive values.

First, a small amount of theory is in order. If a third order determinant has elements a_i and a fixed quantity b is added to each element the resulting determinant would be:

$$\begin{vmatrix} a_1 + b & a_2 + b & a_3 + b \\ a_4 + b & a_5 + b & a_6 + b \\ a_7 + b & a_8 + b & a_9 + b \end{vmatrix}$$

Subtract the second column from the third and the first from the second to obtain

$$\begin{vmatrix} a_1 + b & a_2 - a_1 & a_3 - a_2 \\ a_4 + b & a_5 - a_4 & a_6 - a_5 \\ a_7 + b & a_8 - a_7 & a_9 - a_8 \end{vmatrix}$$

from which it is evident that the value of the altered determinant is

$$D + \lambda b ,$$

where D is the value of the original determinant and λ is the sum of the three minors formed from the second and third columns. Fxpanded and grouped appropriately we obtain

$$\lambda = (a_1a_5+a_5a_9+a_9a_1) + (a_2a_6+a_6a_7+a_7a_2) + (a_3a_4+a_4a_8+a_8a_3)$$
$$- (a_1a_6+a_6a_8+a_8a_1) - (a_3a_5+a_5a_7+a_7a_3) - (a_2a_4+a_4a_9+a_9a_2)$$

This coefficient λ gives the change in the value of the determinant as we add 1 to each of its elements. See [2] for another use.

It should be noted that the groups in λ are the same as those for the positive and negative terms of the determinant expansion and hence any alteration of the arrangement of determinant elements which leaves the expansion unchanged will also be without effect on λ.

An independent investigation shows that the maximum value of λ is 81 when the elements of the determinant are the nine digits, while the value of λ for the determinant giving a maximum of 412 is only 80. Thus, the smaller valued determinant with $\lambda = 81$ will eventually overtake the larger as the elements of the determinants are increased uniformly.

By calculating λ for the largest values given in the table of Bicknell and Hoggatt (Ref. 1, p. 152) λ is found to be 81 for 405 = 630-225a and 630-225c. Adding n to each element of 630-225a, for example, will produce a determinant of value $405 + 81n$; doing likewise for the original maximum determinant of value 412 produces a value of $412 + 80n$. To find when these will be equal, set

$$405 + 81n = 412 + 80n$$

the solution being $n = 7$.

Thus, if we have nine consecutive positive integers beginning with m, the maximum value that can be achieved is $412 + 80m$ if $m \leq 8$; the maximum possible is $405 + 81m$ if $m \geq 8$.

REFERENCES

1. Marjorie Bicknell and Verner E. Hoggatt, Jr., "An Investigation of Nine-Digit Determinants," Mathematics Magazine, 36(1963), 147-152.
2. Marjorie Bicknell and Verner E. Hoggatt, Jr., "Fibonacci Matrices and Lambda Functions," The Fibonacci Quarterly, 1(1963) April, pp. 47-52.

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